Generalization of Quasi Principally Injective S-Acts

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Abstract

The purpose of this paper is to introduce a new kind of generalization of quasi principally injective S-acts over monoids (QP-injective), (and hence generalized quasi injective), namely quasi small principally injective S-acts. Several properties of this kind of generalization are discussed. Some of these properties are analogous to that notion of quasi small principally injective for general modules. Characterizations of quasi small principally injective acts are considered. Conditions are investigated under which acts are inheriting quasi small principally injective property.

Keywords- Quasi Small Principally Injective S-Acts, M-Cyclic Small Subacts, Fully Invariant Small Subacts, Projective Acts, Quasi Principally Injective Acts

I. INTRODUCTION

In this paper we extended concept of the previous works and to generalize new concepts which are: to extend the concept of principally injective acts, to generalize the concept of quasi principally injective acts[1], to establish and extend some new concepts which are dual to quasi principally injective acts [14] and quasi small principally injective acts. Also, we are interested in seeing extend the characterizations and properties of acts remain valid for these previous concepts.

In everywhere of this paper, every S acts is unitary right S acts with zero element 0 which denoted by M0S. We refer the reader to the references ([1],[2],[3],[4],[5],[6],[7],[9],[10],[11],[15]) for basic definitions and terminology relating to S-acts over monoid and injective(projective) acts which are used here.

In [1], the author introduced the concept of quasi principally injective S-act as a generalization of quasi injective. An S-acts N0S is called M-principally injective if for every S-homomorphism from M-cyclic subact of M0 into N0S can be extended to an S-homomorphism from M into N0S (if this is the case, we write N is M-P-injective ). An S-act M0S is called quasi-principally injective if it is M-P-injective, that is every S-homomorphism from M-cyclic subact of M0 into N0S can be extended to S-endomorphism of M0S (for simply QP-injective).

Recently, we adopt the concept of small quasi principally injective S-acts over monoids which represents, on one hand, a generalization of quasi principally injective S-acts and on the other hand representing a generalization of quasi-small principally injective modules [16]. We study their characterizations and properties. Some results on quasi principally injective S-acts [1] and [14] extended to these S-acts.

II. QUASI SMALL PRINCIPALLY INJECTIVE S-ACTS

A. Definition (2.2)

Let M be a right S-acts. A right S-acts N0S is called M-small principally injective (for short MSP-injective) if, every S-homomorphism from an M-cyclic small sub-acts of M0 into N0S can be extended to an S-homomorphism from M0 into N0S. Equivalently, for any endomorphism α of M0 with α(M) is small in M0, every S-homomorphism from α(M) to N0S can be extended to an S-homomorphism from M0 to N0S.

A right S-acts M0S is called quasi small principally injective (for short quasi SP-injective) if it is MSP-injective.

B. Remark and Example (2.2)

1) Let S = (F F)T, where F is a field with T= End (M0S), and M0S = (F F)T. Then M0S is MSP-injective S-acts.

1) Proof

It is easy to show that A = (0 F 0 F) is the only nonzero small M-cyclic sub-acts of M0S. Let α ∈ T such that α (F 0 0 0) ⊆ (0 F 0 0).

Since (1 0 0 0) ∈ M0S, α (1 0 0 0) = (0 x 0 0) for some 0x∈F. Then for each y,z ∈ F...
\[ \begin{align*}
\alpha\left(\begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix}\right) &= \alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix}\right) = \alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}\]

It implies that \( \alpha = 0 \). It implies that \( \xi_T(\ker(\alpha)) = 0 = T\alpha \). This means that \( M_i \) is MSP-injective \( S \)-sets.

2) If \( N \) is small sub-acts of an \( S \)-acts \( M_i \), then \( \bigcup_{i=1}^{n} N_i \) is small sub-acts of \( M_i \).

2) Proof

The proof will be by induction on \( n \). For \( n = 1 \), the assertion holds by the assumption. Assume that \( N = N_1 \cup N_2 \cup \ldots \cup N_{n-1} \) is small sub-acts of \( M_i \). Now, for sub-acts \( B \) of \( M_i \), we have \( N \cup N_B = M_i \). As \( N \) is small, so \( N_B = M_i \) and so \( B = M_i \) since \( N \) is small sub-acts of \( M_i \). Thus, the proof is complete.

3) The \( Z \)-act with multiplication \( Z \) is small-injective but not injective.

C. Proposition (2.3)

Let \( M_i \) and \( N_i \) ( \( 1 \leq i \leq n \)) be a right \( S \)-acts. Then, \( \bigoplus_{i=1}^{n} N_i \) is MSP-injective if and only if \( N_i \) is MSP-injective for each \( i = 1, 2, \ldots, n \).

1) Proof

The necessity is clear. If for the sufficiency we prove the result when \( n = 2 \), then it is enough. Let \( \alpha \in T \) with \( \alpha(M) \) is small in \( M_i \), and \( f: \alpha(M) \to N_i \bigoplus N_2 \) be an \( S \)-homomorphism. Since \( N_i \) ( \( N_2 \) ) are MSP-injective, then there exists \( S \)-homomorphism \( f_1: M_2 \to N_i \) ( \( f_2: M_2 \to N_2 \) ) such that \( f_1 \cdot f = \pi_1 \cdot f \) and \( f_2 \cdot f = \pi_2 \cdot f \) where \( \pi_1, \pi_2 \) is the projection map of \( N_i \) ( \( N_2 \) ) into \( N_i \bigoplus N_2 \) and \( i: \alpha(M) \to M_i \) is the inclusion map. Put \( f_1 \cdot f = \tilde{f} \). Thus \( \tilde{f} \) extends \( f \).

Next corollary represents a generalization of lemma (2.3.11) (1) [12]

D. Corollary (2.4)

Retract of an \( S \)-injective \( S \)-acts is also \( S \)-injective.

The following theorem is a generalization of theorem (2.3) in [16]:

E. Theorem (2.5)

The following conditions are equivalent for projective \( S \)-acts \( M_i \):

1) Every \( M \)-cyclic small sub-act of \( M_i \) is projective.

2) Every factor of an \( S \)-injective \( S \)-act is \( S \)-injective.

3) Every factor of an injective \( S \)-act is \( S \)-injective.

1) Proof

(1 \( \Rightarrow \) 2) Let \( A_i \) be an \( S \)-injective \( S \)-acts and \( \sigma(M) \) be \( M \)-small sub-acts in \( M_i \). Let \( \alpha: \sigma(M) \to A_i / \rho \) be an \( S \)-homomorphism. Then by (1), there exists \( S \)-homomorphism \( \beta: \sigma(M) \to A_i \) such that \( \pi_1 \beta = \alpha \) where \( \pi_1: A_i = A_i / \rho \) is the natural epimorphism. Since \( A_i \) is MSP-injective, so \( \beta \) can be extended to \( S \)-homomorphism \( f: M_i \to A_i \). Put \( \varphi = \pi f \), so \( \varphi \) is the extension of \( \alpha \) to \( M_i \).

(2 \( \Rightarrow \) 3) Assume that \( E \) is injective \( S \)-acts and \( E / \rho \) is the factor of \( E \). Since every injective \( S \)-acts is \( S \)-injective acts, so \( E \) is \( S \)-injective acts. Then, by (2) \( E / \rho \) is \( S \)-injective acts.

(3 \( \Rightarrow \) 1) Let \( \alpha(M) \) be \( M \)-cyclic small sub-acts of \( M_i \), and \( f: A_i \to B_i \) be an \( S \)-epimorphism, where \( A_i \) and \( B_i \) be two \( S \)-acts. Then \( B_i \cong A_i / \rho \), where the congruence \( p = \ker(f) \). Let \( g: \alpha(M) \to B_i \). Since every \( S \)-acts can be embedding in injective acts by corollary (1.6) [8, p.186], so embed \( A_i \) in injective acts \( E \). Then \( B_i \cong A_i / \rho \) is a sub-acts of \( E / \rho \). By (3) \( g \) extends to \( \tilde{g}: M_i \to E / \rho \). As \( M_i \) is projective, so \( \tilde{g} \) can be lifted to \( \sigma: M_i \to E \). It is obvious that \( \sigma(\alpha(M)) \subset A_i \). Put \( \sigma i = \beta \), where \( i \) is the inclusion map of \( \alpha(M) \) into \( M_i \). This means that \( \beta: \alpha(M) \to A_i \). Thus \( g \) lifted to \( \beta \).

For the endomorphism monoid, we have the following proposition:

F. Proposition (2.6)

Let \( M_i \) be a right \( S \)-acts and \( T = \text{End} (M_i) \). Then, the following equivalent are equivalent:

1) \( M_i \) is quasi small principally-injective.

2) \( \ell_T(\ker(\alpha)) = T\alpha \) for all \( \alpha \in T \) with \( \alpha(M_i) \) small in \( M_i \).

3) \( \ker(\alpha) \subseteq \ker(\beta) \) where \( \alpha, \beta \in T \) with \( \alpha(M_i) \) small in \( M_i \), implies \( T\beta \subseteq T\alpha \).

4) \( \ell_T(\ker(\alpha) \cap \beta(M)) = \ell_T(\beta(M) \times \beta(M) / \beta(M)) \cup T\alpha \) for \( \alpha, \beta \in T \) with \( \alpha(M_i) \) small in \( M_i \).

5) If \( \sigma: \alpha(M_i) \to M_i, \alpha \in T \) with \( \alpha(M_i) \) small in \( M_i \), then \( \sigma T \subseteq T \alpha \).

1) Proof

(1 \( \Rightarrow \) 2) Let \( \alpha \in T \) with \( \alpha(M) \) is small in \( M_i \) and let \( \beta \in \ell_T(\ker(\alpha)) \). Then \( \ker(\alpha) \subseteq \ker(\beta) \), so there exists an \( S \)-homomorphism \( \sigma: \alpha(M) \to N_i \) such that \( \sigma \alpha = \beta \). Since \( \alpha(M) \) is small in \( M_i \), and \( M_i \) is quasi small \( P \)-injective, so there exists an \( S \)-homomorphism \( \sigma: M_i \to M_i \) such that \( \sigma i = \sigma \), where \( i: \alpha(M) \to M_i \) is the inclusion map. Therefore \( \beta = \sigma \alpha \in T \alpha \). For the other direction let \( \beta \in T \alpha \), then \( \beta = \sigma \alpha \) for some \( \sigma \in T \). For each \( s,t \in S \) with \( ms = mt \), we have \( \alpha(ms) = \alpha(mt) \) and then \( \sigma(\alpha(ms)) = \alpha(\sigma(ms)) \). Thus, it is implies that \( \beta(ms) = \beta(mt) \), and so \( \beta \in \ell_T(\ker(\alpha)) \).
(2→1) Let $\alpha \in T$ with $\alpha(M)$ is small in $M$, and $\sigma : \alpha(M) \to M$ be an $S$-homomorphism. Then $\sigma \in T$ and $\sigma \in \ell_T(\ker(\alpha))$. By assumption, $\sigma = f a$ for some $f \in T$. This implies that $M_1$ is quasi small $P$-injective.

(2→3) Let $\ker(\alpha) \subseteq \ker(\beta)$, where $\alpha, \beta \in T$ with $\alpha(M)$ is small in $M$. Then, $\ell_T(\ker(\beta)) \subseteq \ell_T(\ker(\alpha))$. Since $T \beta \subseteq \ell_T(\ker(\beta))$ and by (2) we have $\ell_T(\ker(\alpha)) = T \alpha$, so $T \beta \subseteq T \alpha$.

(3→4) Let $\sigma \in \ell_T(\ker(\alpha) \cap (\beta(M) \times \beta(M)))$. We claim that $\ker(\beta) \subseteq \ker(\beta)$, for this let $m, m_1 \in \ker(\beta)$, so $\alpha(m_1) = \alpha(m)$. This implies that $\beta(m_1), \beta(m) \in (\ker(\beta) \cap (\beta(M) \times \beta(M)))$. Then $\alpha(\beta(m_1)) = \alpha(\beta(m))$. Thus $(m_1, m) \in \ker(\beta)$. By (3), we have $T \sigma \beta \subseteq T \alpha \beta$ and $\sigma = u \sigma \beta$ for some $u \in T$. This means that there is $M$ such that $\sigma = \sigma \beta$ for each $\sigma, \beta \in T$. In particular $\sigma = u \sigma \beta$. Thus $\sigma \in \ell_T(\beta(M) \times \beta(M))$. Conversely, let $\sigma \in \ell_T(\beta(M) \times \beta(M))$. Then $\sigma \beta = \sigma \beta \alpha$. This means that $\sigma = \sigma \beta$ for some $u \in T$.

(4→5) Put $\beta = I_M$, identity map of $M$, then we have $\sigma \in \ell_T(\ker(\alpha)) = \ell_T(\beta(M) \times \beta(M) = \ell_T(\beta(I_M) \times \beta(I_M)) = \ell_T(\beta(I_M) \times (I_M))T \alpha = T \alpha$.

(5→1) It is obvious.

G. Corollary (2.7)
The following conditions are equivalent for monoid $S$:
1) $S$ is SP-injective.
2) $\ell_T(\gamma_2(a)) = S a$ for all $a \in S$ with $aS$ is small in $S_n$.
3) $\gamma_3(b) \subseteq \gamma_3(a)$, where $a, b \in T$ with $aS$ is small in $S_n$, implies $S a \subseteq S b$.
4) $\ell_T(b \cap \gamma_3(a) \times \gamma_3(a)) = \ell_T(b \times b) S a$ for $a \in S$ with $aS$ is small in $S_n$.
5) If $\alpha : aS \rightarrow S a, \alpha \in S$ with $aS$ is small in $S_n$, then $\alpha \sigma \in S$.

H. Proposition (2.8)
Retract of a quasi-small principally injective $S$-act is quasi-small principally injective.

1) Proof
Let $M_1$ be quasi small principally injective $S$-acts and $N$ is a retract of $M_1$. Let $\alpha \in T = \text{End}(N)$ with $\alpha(N)$ be $N$-cyclic small sub-acts of $N$ and then in $M_1$ (this means that $\alpha(N)$ is $M$-cyclic small sub-acts of $M_1$) by lemma 3.4 in [13] and let $f : \alpha(N) \to N$ be $S$-homomorphism. Since $M_1$ is quasi small principally injective $S$-acts, so there exists $g : M_1 \to M_1$ such that $g \beta |_{\alpha(N)} = j_N f$ where $j_N$ is the injection of $N$ into $M_1$. Let $\overline{g} = g|_N : N \to N$. Thus, it is clear that $\overline{g}$ is extension of $f$ and $N$ is quasi small principally injective $S$-acts.

I. Proposition (2.9)
Let $M_1$ be quasi SP-injective $S$-acts and $\alpha_1 \in T$ with $\alpha_1(M)$ is $M$-cyclic small sub-acts of $M_1 (1 \leq i \leq n)$
1) If $T \alpha_1 \oplus T \alpha_2 \oplus \ldots \oplus T \alpha_n$ is direct, then any $S$-homomorphism $\beta : \alpha_1(M) \cup \alpha_2(M) \cup \ldots \cup \alpha_n(M) \to M$ has an extension in $T$.
2) If $\alpha_1(M) \oplus \alpha_2(M) \oplus \ldots \oplus \alpha_n(M)$ is direct, then $T(\alpha_1, \alpha_2, \ldots, \alpha_n) = T \alpha_1 \cup T \alpha_2 \cup \ldots \cup T \alpha_n$.

1) Proof
Let $\beta : \alpha_1(M) \cup \alpha_2(M) \cup \ldots \cup \alpha_n(M) \to M$ be an $S$-homomorphism with $\alpha_1(M)$ is $M$-cyclic small sub-acts of $M_1$. Since $M_1$ is quasi SP-injective S-acts, so there exists an $S$-homomorphism $\alpha_1 : M_n \to M$, such that $\alpha_1(M) = \alpha_1(m)$ for all $m \in M_1$. Since $\alpha_1(M) \cup \alpha_2(M) \cup \ldots \cup \alpha_n(M)$ is small sub-acts of $M_1$ by remark and example (2.2)(2), so $\beta$ can be extended to $\beta : M_n \to M$ such that for any $m_1 \in M_1$, $\beta(\cup_{i=1}^n \alpha_i(m)) = \beta(\cup_{i=1}^n \alpha_i(m))$. This implies that $\cup_{i=1}^n \alpha_i = \cup_{i=1}^n \beta_i$. Since $T \alpha_1 \oplus T \alpha_2 \oplus \ldots \oplus T \alpha_n$ is direct, so $\beta_1 = \sigma_1 \alpha_1$ for all $(1 \leq i \leq n)$. Thus $\beta$ is an extension of $\beta$.

Let $\beta \beta_1 \alpha_1 \beta_2 \alpha_2 \ldots \beta_n \alpha_n \in T \alpha_1 \cup T \alpha_2 \cup \ldots \cup T \alpha_n$. Define $\sigma_i(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sigma_i(M)$ for each $i$ by $\alpha_i((\alpha_1, \alpha_2, \ldots, \alpha_n)) = \alpha_i(M)$ for all $m \in M$. Since $\alpha_i(M) \oplus \alpha_2(M) \oplus \ldots \oplus \alpha_n(M)$ is direct, so $\sigma_i$ is well-defined. For this let $\alpha_1(M) = (\beta_1, \beta_2, \ldots, \beta_n)$ for each $\alpha_1 \in T$ and $m \in M$, this implies that $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n)$, then $\alpha_1(m) = \beta_1(m)$. Thus $\sigma_i((\alpha_1, \alpha_2, \ldots, \alpha_n)) = \sigma_i(\beta_1, \beta_2, \ldots, \beta_n)$ for each $i$. As $\cup_{i=1}^n \alpha_i(M)$ is small sub-acts of $M_1$ by remark and example (2.2)(2), and since $M_1$ is quasi SP-injective S-acts, so there exists an $S$-homomorphism $\sigma_i$ in $T$ which is extension of $\beta_i$. Then $\alpha_1 = \sigma_i(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sigma_i(M) = \sigma_i(T(\alpha_1, \alpha_2, \alpha_n))$. This implies that $T \alpha_1 \cup T \alpha_2 \cup \ldots \cup T \alpha_n = T(\alpha_1, \alpha_2, \ldots, \alpha_n)$. The reverse inclusion is always holds.

J. Proposition (2.10)
Let $M_1$ be quasi SP-injective $S$-acts with $T = \text{End}(M_1)$, and let $A$ be small sub-act of $M_1$. Let $\bigoplus_{i=1}^n \alpha_i(M)$ be direct sum of small $M$-cyclic small sub-act of $M_1$. Then for any small $\Lambda$ of $M_1$, we have $\Lambda \cap \bigoplus_{i=1}^n \alpha_i(M) = \bigoplus_{i=1}^n (\Lambda \cap \alpha_i(M))$.

1) Proof
Let \( x \in \bigoplus_{i=1}^{n} (A \cap \alpha_i(M)) \), then there exists \( j \in \{1, 2, \ldots, n\} \), such that \( x \in A \cap \alpha_i(M) \) which implies that \( x \in A \) and \( x \in \alpha_i(M) \) for some \( j \in I \), so \( x \in A \cap \bigoplus_{i=1}^{n} \alpha_i(M) \). Then \( \bigoplus_{i=1}^{n} (A \cap \alpha_i(M)) \subseteq A \cap \bigoplus_{i=1}^{n} \alpha_i(M) \). Conversely, let \( a \in A \cap \bigoplus_{i=1}^{n} \alpha_i(M) \) which implies that \( a \in A \) and \( a \in \bigoplus_{i=1}^{n} \alpha_i(M) \). So there exists \( j \in I \) such that \( a \in \alpha_j(M) \). Let \( \pi_j : \bigoplus_{i=1}^{n} \alpha_i(M) \rightarrow \alpha_j(M) \) be the projection, then take \( \sigma = \pi_j \alpha_j(M) : \alpha_j(M) 
rightarrow \alpha_j(M) \). Let \( i_1, i_2 \) be the inclusion maps of \( \alpha_i(M) \) and \( \alpha_j(M) \) into \( M_i \), respectively. Since \( \alpha_j(M) \) is small sub-acts of \( M_i \) and \( M_j \) is quasi SP-injective acts, so by (1) of proposition (2.9), \( \sigma \) can be extended to \( S \)-homomorphism \( \beta : M_i \rightarrow M_j \) that is there exists \( \beta \in T \), so \( \beta \) extends \( \pi_j \). Thus for \( a \in \alpha_j(M) \), we have \( \alpha_j(m) = \pi_j(a) = \beta(a) = \sigma(a) \). Then, \( a \in \bigoplus_{i=1}^{n} (A \cap \alpha_i(M)) \) and \( A \cap \bigoplus_{i=1}^{n} \alpha_i(M) \subseteq \bigoplus_{i=1}^{n} (A \cap \alpha_i(M)) \).

### III. Conclusions

From this work, we can put a highlight on some important points:

First: We have illustrated in the theorem (2.3) when the direct sum of finite \( M \)-small principally injective is also \( M \)-small principally injective and in corollary (2.4), clarified when the direct summand (retract) of \( M \)-small principally injective, is also \( M \)-small principally injective.

Second: In theorem (2.5), we found the relationship between the factor of injective and \( M \)-small principally injective acts under projective condition. Besides, we found when \( M \)-cyclic subact of projective is projective?

Third: Proposition (2.6), corollary (2.7), and proposition (2.9) demonstrated the relationship between endomorphism monoid and acts under \( M \)-small principally injective property.

For the future work, one can extend this work by taking subacts as small finitely generated.

### References