An Efficient Implementation of Tower of Hanoi using Gray Codes

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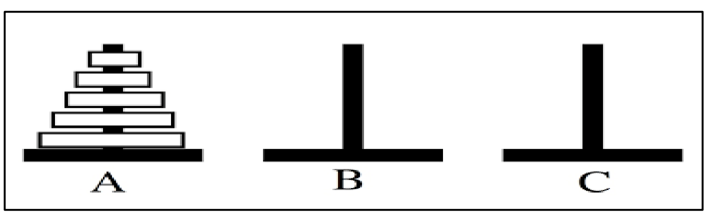
Abstract

The Tower of Hanoi Puzzle finds its applications ranging from robotics to psychological research. This puzzle is a classic case of recursive algorithm in programming. However, this puzzle can also be implemented using iterative programming, by using binary codes or gray codes. Various applications require an optimized solution for this puzzle. In this paper, an efficient implementation of Tower of Hanoi using Gray codes for 'n' disks and three rods is presented. This focuses only on minimizing storage and reducing running time as required by many applications. The proposed implementation using Gray code system consumes lesser memory and slightly reduced running time compared to the conventional recursive methodology.
Keyword- Tower of Hanoi; Gray Codes; Recursion; Non-recursive algorithm

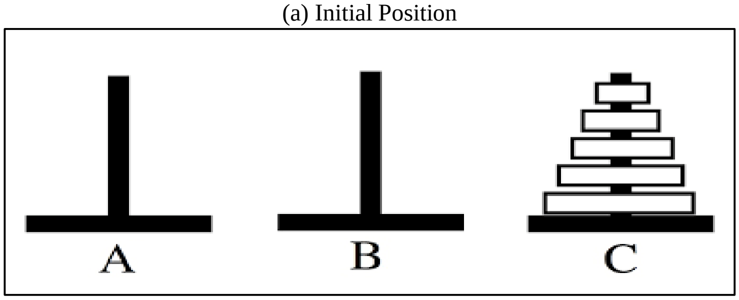
I. INTRODUCTION

The Tower of Hanoi, a mathematical game or puzzle, was invented by E. Lucas in 1883 [1, 3]. It consists of three rods, and a number of disks of different sizes which can slide onto any rod. The disks in a stack are arranged in ascending order of size on one rod, the smallest at the top, thus making a conical shape [4]. The objective of this puzzle is to move the entire stack from one rod to another rod using an intermediate rod, following the three rules: (i) Only one disk can be moved at a time, (ii) Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e., a disk can only be moved if it is the uppermost disk on a stack and (iii) No larger disk may be placed on top of a smaller disk. Figure 1 represents the initial and final positions of Tower of Hanoi. The minimum number of moves required to solve a Tower of Hanoi puzzle is (2n – 1), where n is the number of disks. With three disks, the puzzle can be solved in seven moves. The problem is isomorphic to finding a Hamiltonian path on an n-hypercube [5, 6].

The Tower of Hanoi puzzle is used in different applications [7]. It is used as a backup rotation scheme for performing computer data backups where multiple tapes or media are involved. It provides a good standardized test bed to evaluate integration of high-level reasoning capabilities of robots together with their manipulation and perception aspects. It is frequently used in psychological research on problem solving. The Tower of Hanoi is also used as a test by neuropsychologists trying to evaluate frontal lobe deficits [7].



(a) Initial Position



(b) Final Position

Fig. 1: Tower of Hanoi Puzzle

The rest of the paper is organised as follows: Section II presents the existing recursive methology and its pitfalls. Section III presents the proposed solution using Gray codes. Section IV discusses on the performance of the proposed solution. Section V gives the concluding remarks and future scope of this work.

II. RELATED WORK

Recursive algorithm is the preferred solution for this puzzle. There are also other solutions which are based on non-recursive algorithms, binary codes and gray codes. Iterative solution alternates the moves between the smallest disk and a non-smallest disk. Recursive solution breaks the problem into a collection of smaller problems and further breaking those problems down into even smaller problems until a solution is reached [1, 4, 7]. This is carried out by the following the steps:

1) move n−1 disks from source peg A to intermediate peg B,
2) move disk n from peg A to the destination peg C and
3) move n−1 disks from peg B to peg C such that they are placed on disk n.

```
Start
Procedure Hanoi(disk, source, dest, aux)
  if disk == 0 then
    move disk from source to dest
  else
    Hanoi(disk - 1, source, aux, dest)
    move disk from source to dest
    Hanoi(disk - 1, aux, dest, source)
  end if
    end Procedure
STOP
```

Fig. 2: Algorithm for the Recursive Implementation

The algorithm for recursive implementation is given in Figure 2. The recursive solution is the easiest to comprehend, but fails to accommodate as number of disks increases.

In the binary solution, the disk positions are determined directly from the binary (base 2) representation of the move number (the initial state being move 0, with all digits 0, and the final state being, with all digits 1). The proposed solution using Gray code gives an alternative way of solving the puzzle [2, 4, 7]. All these solutions aim at minimizing the number of moves.

III. METHODOLOGY

The aim of this work is to implement Tower of Hanoi puzzle using Gray codes with lesser memory and lesser running time compared to the recursive algorithm for three rods and 'n' disks. In the Gray code system, numbers are expressed in a binary combination of 0s and 1s, but rather than being a standard positional numeral system, Gray code operates on the premise that each value differs from its predecessor by exactly one bit changed [2,4,7].

1. **for** step no i

    1.1 Compute i^{th} and $(i+1)^{th}$ grey number.

    1.2 XOR the two numbers and note which bit changes.

    1.3 Let the bit change occur at position j

        1.3.1 **if** j is equal to n, GOTO 2

    1.4 Compute the FROM_PEG and TO_PEG using the formula

        FROM_PEG = (i & (i-1)) % 3

        TO_PEG = ((i | i-1) +1) % 3

    1.5 **if** the number of disks is even then swap the peg 2 with peg 3.

    1.6 Move disk j from FROM_PEG to TO_PEG.

    1.7 GOTO step 1.1

2. Stop.

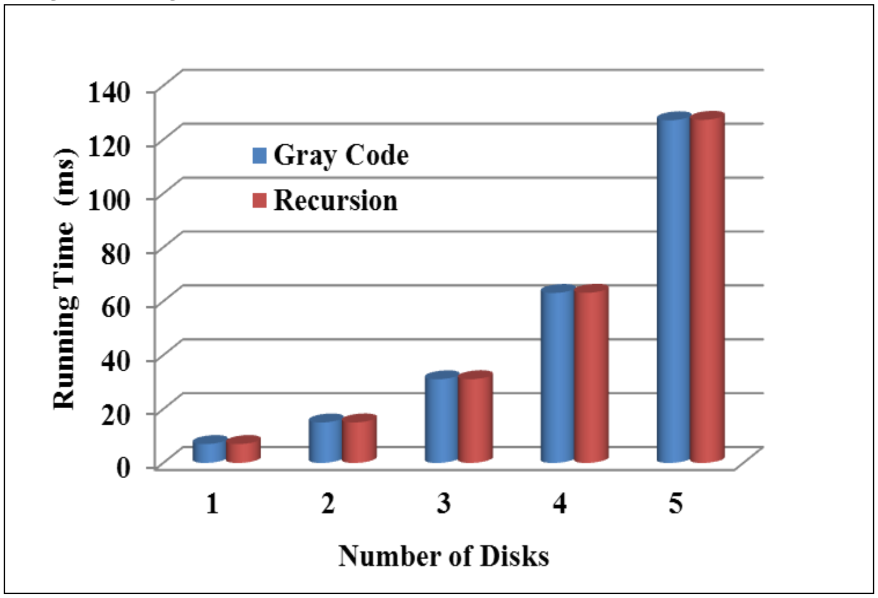
Fig. 3: Algorithm for the Gray Code Implementation

The number of bits present in Gray code is important, and leading zeros are not optional, unlike in positional systems. If one counts in Gray code of a bit size equal to the number of disks in Tower of Hanoi, begins at zero, and counts up, then the bit changed each move corresponds to the disk to move, where the least-significant-bit is the smallest disk and the most-significant-bit is the largest. Counting moves from 1 and identifying the disks by numbers starting from 0 in order of increasing size, the ordinal of the disk to be moved during move m is the number of times m can be divided by 2 [7]. This implementation is for n disks and 3 rods. The algorithm for the Gray code methodology is presented in Figure 3.

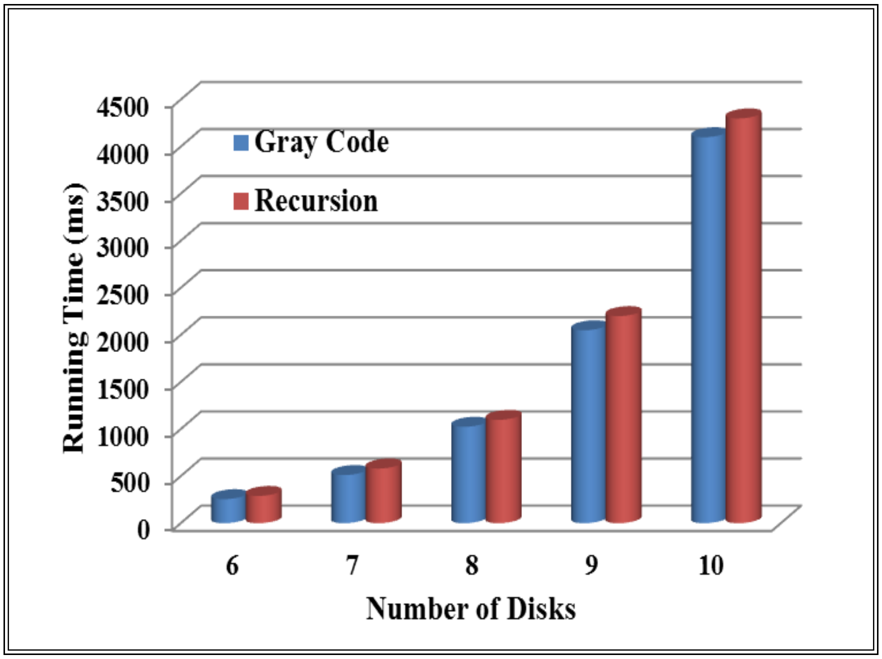
IV. DISCUSSION

The implementation of tower of Hanoi puzzle using Gray code is analysed based on: (i) running time and (ii) memory. Its performance is compared with the conventional recursive algorithm.

The impact of number of disks on running time is presented as graph in Figure 4. In Figure 4 (a), the number of disks is less than or equal to 5 and in Figure 4 (b), the number of disks ranges from 6 to 10. From the Figure 4(a), it is evident that the running time for disks lesser than or equal to 5 is almost the same for Gray Code implementation and recursive method. But as the number of disks increases the running time of the recursive method is slightly higher than the Gray code implementation. It is seen from Figure 4 (b) that the running time is higher for recursive method for more than 6 disks.



(a) Less than or equal to 5 disks



(b) 6 to 10 disks

Fig. 4: Impact of number of disks on running time

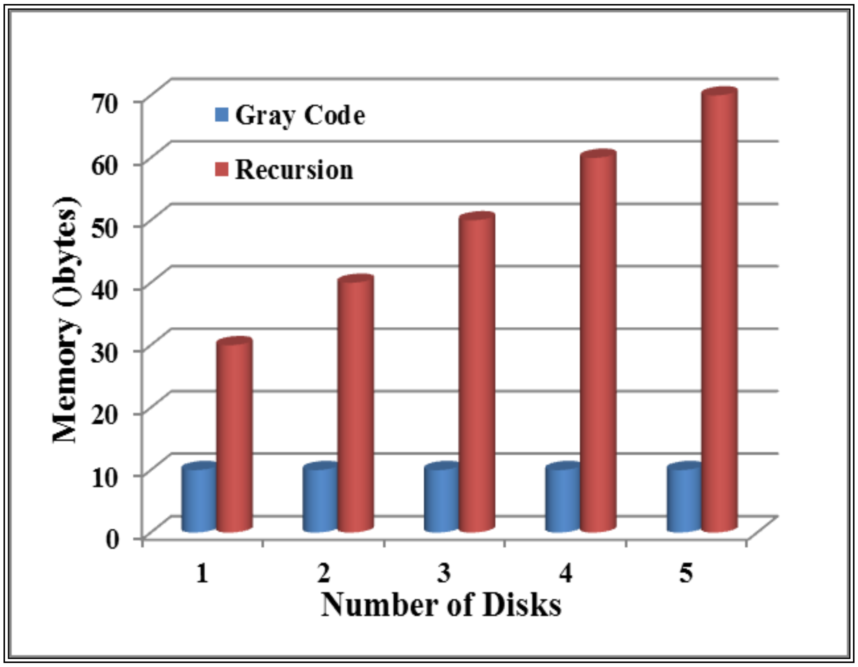


Fig. 5: Impact of number of disks on memory

The impact of number of disks on memory requirement for the recursive method increases as the number of disks increases. But the Gray code implementation consumes lesser memory irrespective of the number of disks. This is evident from Figure 5.

The number of moves m_n required to solve the puzzle of 'n' disks on three rods is given by the recurrence relation

$$M_n = 2M_{(n-1)} + 1 \qquad (1)$$

With $M_1=1$ and solving the recurrence relation we get,

$$M_n = 2^n - 1 \qquad (2)$$

For three rods, the proof that the above solution is minimal can be achieved using the Lucas correspondence which relates Pascal's triangle to the Hanoi graph. While algorithms are known for transferring disks on four rods, none has been proved minimal. As the number of disks is increases (for three rods), an infinite sequence is obtained, the first few of them are presented in Table 1. This is exactly the binary carry sequence plus one. The number of disks moved after the kth step is the same as the element which needs to be added or deleted in the kth value of the Ryser formula [2, 8-10].

Number of Disks (n)	*Sequence of Moves (S_n)*
1	*1*
2	*1, 2, 1*
3	*1, 2, 1, 3, 1, 2, 1*
4	*1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1*

Table 1: Sequence of Moves

A Hanoi graph can be constructed whose graph vertices correspond to legal configurations of n towers of Hanoi, where the graph vertices are adjacent if the corresponding configurations can be obtained by a legal move. The puzzle itself can be solved using a binary Gray code. Poole and Rangel-Mondragón give Wolfram Language routines for solving the Hanoi towers problem [2, 8, 11-13]. Poole's algorithm works for an arbitrary disk configuration, and provides the solution in the fewest possible moves [8, 11-12].

V. CONCLUSION

This paper presents an efficient implementation of Tower of Hanoi using Gray Codes for three rods and n disks. The proposed solution focuses at reducing the memory requirement and running time compared to conventional recursive methodology. As a future work, this work can be extended to provide solution to the tower of Hanoi puzzle greater than three rods.

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