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On Triangular Sum Labelings of Graphs

Shankaran P
Department of Mathematics
N.M.A.M. Institute of Technology, Nitte-574 110, Karnataka, India

Abstract

Let G = (V,E) be a (p, q)-graph. A graph G is said to admit a triangular sum labeling, if its vertices can be labeled by non-negative integers so that the values on the edges, obtained as the sum of the labels of their end vertices, are the first q triangular numbers. In this paper, we obtain a necessary condition for an eulerian graph to admit a triangular sum labeling and show that some classes of graphs admit a triangular sum labeling. Also we show that some classes of graphs can be embedded as an induced subgraph of a triangular sum graph.
Keywords- Triangular Numbers, Triangular Sum Labeling/Graphs, Dutch Windmill, Locally Finite Tree
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I. INTRODUCTION

Several practical problems in real-life situations have motivated the study of labelings of graphs which are required to obey a variety of conditions depending on the structure of graphs. There is an enormous amount of literature built upon several kinds of labelings over the past four decades or so and for a survey of results on labelings we refer to [2].
For various graph theoretical notations and terminology we refer to Harary [3] and West [10].
Definition 1.1. [4]. When n copies of K_3 share a common vertex, then the resulting graph is called a dutch windmill, denoted as DW(n).

II. MAIN RESULTS

Definition 2.1. Let G = (V, E) be a graph with p vertices and q edges. Let T_i be the i^{th} triangular number given by $T_i = i(i+1)/2$ (see [1]). A triangular sum labeling of a graph G is a one-to-one function $f: V(G) \to N$ (where N is the set of non-negative integers) that induces a bijection $f^+: E(G) \to \{T_1, T_2, \ldots, T_q\}$ defined by $f^+(uv) = f(u)+f(v)$, $\forall$ $e = uv \in E(G)$. The graph which admits such a labeling is called a triangular sum graph.
We adopt the following notation throughout this paper.

$$f(G) = \{ f(u)/u \in V(G)\} \text{ and}$$
$$f^+(G) = \{ f^+(e)/e \in E(G)\}.$$

Example 2.2. The path Pn = ($v_1, v_2, \ldots, v_n$) is a triangular sum graph.
The function $f: V(P_n) \to N$ defined by $f(v_1) = 0$ and $f(v_i) = T_{i-1} - f(v_{i-1})$, $2 \le i \le n$ is such that $f^+(P_n) = \{T_1, T_2, \ldots, T_{n-1}\}$. Triangular sum labeling of P_6 is given in Figure 1.

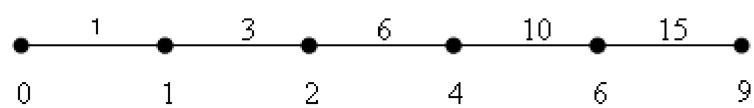

Fig. 1: Triangular sum labeling of P_6

Example 2.3. The star $K_{1,n}$ is a triangular sum graph. If $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ with deg(v) = n, then f defined by $f(v) = 0$ and $f(v_i) = T_i$ is a triangular sum labeling of $K_{1,n}$.
Theorem 2.4. If G is an eulerian (p,q)-graph admitting a triangular sum labeling, then $q \equiv 0,2,3,4,5,6,7,8,9,10$ or $11 \pmod{12}$.
Proof. Let G be an eulerian (p,q)-graph admitting a triangular sum labeling f. Since an eulerian graph can be decomposed into edge-disjoint cycles, the sum of the edge values is always even. Further the sum of the first q triangular numbers is

$$\frac{q(q+1)(q+2)}{6}.$$

Hence q(q+1)(q+2) is a multiple of 12 and hence the result follows
Theorem 2.5. Any tree obtained from a star $K_{1,n}$ by extending each edge to a path is a triangular sum graph.

Proof. Let $K_{1,n}$ be the given star with the root vertex $u_{1,0}$ and T be a rooted tree obtained by replacing each edge by a path (u_{10}, $u_{i1}, u_{i2},\ldots, u_{i\,mi}$). Note that the tree T contains $\sum_{i=1}^{n} m_i$ edges and n paths each of length at least two.

We consider three cases.

Case 1. All the m_i's are distinct.

Let $m_1 > m_2 > \ldots > m_n \geq 2$ and r_j be the number of vertices at j^{th} level, $1 \leq j \leq m_1$.

Clearly $1 \leq r_{m_1} \leq r_{m_1-1} \leq \ldots \leq r_2 \leq r_1 = n$. Define a labeling $f:V(T) \to N$ by

$$f(u_{1,0}) = 0$$
$$f(u_{i,1}) = T_i,\ 1 \leq i \leq n \text{ and}$$
$$f(u_{i,j}) = T_{i+k} - f(u_{i,j-1}),\ 1 \leq i \leq r_j,\ 2 \leq j \leq m_1,\ k = \sum_{t=1}^{j-1} r_t$$

Then one can easily check that the vertex labels are all in increasing order and $f^+(T) = \{T_1, T_2, \ldots, T_q\}$. Hence the tree T is a triangular sum graph.

Case 2. All the m_i's are equal.

Let $m_i = m$, $1 \leq i \leq n$. Define a mapping $f:V(T) \to N$ by

$$f(u_{1,0}) = 0,$$
$$f(u_{i,1}) = T_i,\ 1 \leq i \leq n \text{ and}$$
$$f(u_{i,j}) = T_{i+(j-1)n} - f(u_{i,j-1}),\ 1 \leq i \leq n,\ 2 \leq j \leq m.$$

Then one can see that the vertex labels are all in increasing order and $f^+(T)$ is the set of triangular numbers $T_1, T_2, \ldots, T_q$.

Case 3. $m_1 \geq m_2 \geq \ldots \geq m_n \geq 2$ and not all m_i's are equal.

Let $1 \leq r_{m_1} \leq r_{m_1-1} \leq \ldots \leq r_2 = r_1 = n$, where r_j be the number of vertices at j^{th} level. Define,

$$f(u_{1,m1}) = 0,$$
$$f(u_{1,\,m1-j}) = T_j - f(u_{1,\,m1-j+1}),\ 1 \leq j \leq m_1$$
$$f(u_{i,1}) = T_{m_1+i-1} - f(u_{1,0}),\ 2 \leq i \leq n$$
$$f(u_{i,j}) = T_{m_1+k+i-j} - f(u_{i,j-1}),\ 2 \leq i \leq r_j,\ 2 \leq j \leq m_2 \text{ and } k = \sum_{t=1}^{j-1} r_t$$

Then $f(u_{i+1,j}) > f(u_{i,j})$, $\forall$ i,j and $f^+(T) = \{T_1, T_2, \ldots, T_q\}$

Example 2.6. The tree given in Figure 2a is obtained by replacing each edge of the star $K_{1,3}$ by paths of different length and the tree given in Figure 2b is obtained by replacing each edge of the star $K_{1,4}$ by paths of equal length. The triangular sum labelings of the trees are also given.

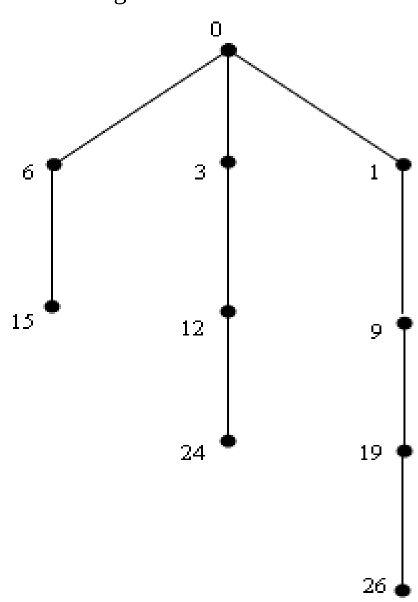

Fig. 2 (a)

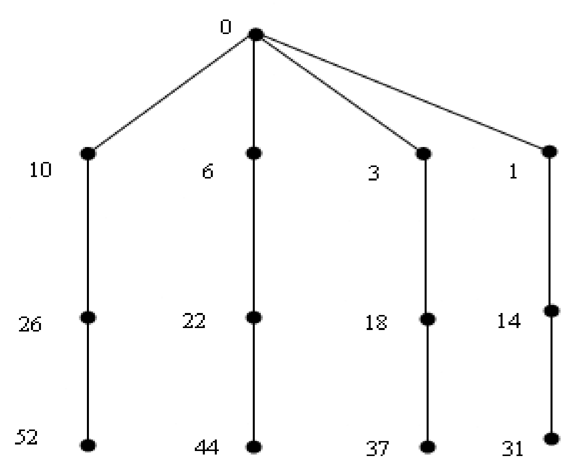

Fig. 2 (b)

Theorem 2.7. The lobster T obtained by joining the centres of k copies of a star to a new vertex w is a triangular sum graph.

Proof. Let T be the lobster obtained by joining the centre of k stars $K_{1,n}$. Denote the root vertex of i^{th} star $K_{1,n}$ as w_i, i=1,2,…,k, the pendant vertices of i^{th} star as $v_{i,j}$, i=1,2,…,k, j=1,2,…,n. Note that T contains nk+k edges. Define a labeling f:V(T)→ N by

$$f(w) = 1,$$
$$f(w_i) = T_i - 1,\ 1 \le i \le k \text{ and}$$
$$f(v_{i,j}) = T_{k+j+m} - f(w_i),\ 1 \le i \le k, 1 \le j \le n,\ m = (i-1)n.$$

Since $f(v_{i,j}) > f(v_{i,j+1})$ for some i, j, we have $f(v_{i,j}) + f(w_i) > f(v_{i,j+1}) + f(w_i)$ and one can see that $f^+(T) = \{T_1,T_2,\ldots,T_q\}$. Thus T is a triangular sum graph.

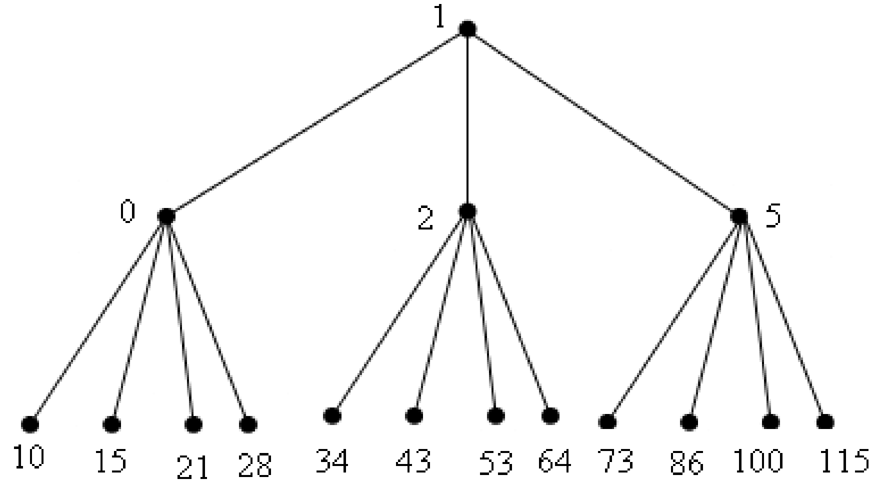

Fig. 3: An example of a triangular sum labeling of a lobster

Theorem 2.8. The complete n-ary tree Tm of level m is a triangular sum graph.

Proof. Denote the root vertex of T_m as u_0; the vertices of level 1 as $u_{11},u_{12},\ldots, u_{1n^1}$; that of level 2 as $u_{21},u_{22},\ldots,u_{2n^2}$;…; the vertices of level m as $u_{m1},u_{m2},\ldots, u_{mn^m}$. Note that T_m contains $\sum_{i=1}^{m} n^i$ edges. Define the map $f:V(T_m) \to N$ by

$$f(u_0) = 0,$$
$$f(u_{1,j}) = T_j,\ 1 \le j \le n^1$$
$$f(u_{i,j}) = T_{t+j} - f(u_{i-1,r}),\ 2 \le i \le m,\ 1 \le j \le n^i, \text{ where } t = \sum_{k=1}^{i-1} n^k$$
$$r = 1 \text{ if } 1 \le j \le n$$
$$r = 2 \text{ if } n+1 \le j \le 2n$$
$$r = 3 \text{ if } 2n+1 \le j \le 3n$$
$$\ldots\ \ldots\ \ldots\ \ldots$$
$$r = n^{m-1} \text{ if } n^m-n+1 \le j \le n^m$$

Since $f(u_{i+1,j+1}) > f(u_{i+1,j})$, it follows that $f(u_{i,r}) + f(u_{i+1,j})$ are all distinct and $f^+(T_m)=\{T_1,T_2,\ldots,T_q\}$.

Example 2.9. An infinite graph G is said to be locally finite if every vertex of G has finite degree. We now give an example of a locally finite tree which admits a triangular sum labeling. Consider the locally finite tree given in Figure 4. Let {ui: i ≥1} be the vertices of the infinite path in T and let vi be a pendant vertex adjacent to ui. Then f defined by

$$f(u_1) = T_1,$$
$$f(u_2) = 0,$$
$$f(u_i) = T_{2i-2} - f(u_{i-1}),\ i \ge 3,$$
$$f(v_1) = T_2 - 1 \text{ and}$$
$$f(v_i) = T_{2i-1} - f(u_i),\ i \ge 2$$

is a triangular sum labeling of G.

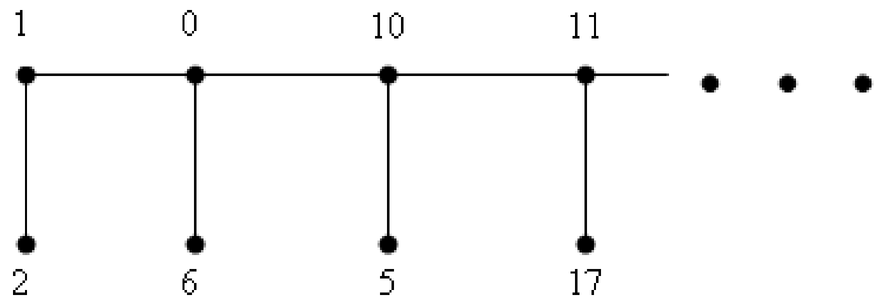

Fig. 4: Locally finite tree T

We propose the following conjecture.
Conjecture 2.10. All trees are triangular sum graphs.
Theorem 2.11. The complete graph Kn is triangular sum if and only if $n \le 2$.
Proof. Trivially K_1 and K_2 are triangular sum graphs.
Assume $n \ge 3$. Assign the numbers 0 and 1 to any two vertices of K_n. To get the next triangular number 3, we have to assign the numbers either 2 or 3 to a third vertex. In both cases, we get a non-triangular number 2 or 4 as an edge label, and hence K_n is not a triangular sum graph if $n \ge 3$.
Theorem 2.12. The dutch windmill DW (n) is not a triangular sum graph.
Proof. Denote the root vertex of the dutch windmill DW(n) as u and the remaining two vertices of i^{th} triangle as v_{2i-1} and v_{2i} for $1 \le i \le n$. Suppose DW(n) admits a triangular sum labeling f.
Case 1. f(u) = 0.
Then all v_i's, i=1,2,…,2n must be assigned triangular numbers. However, $1+T_i \neq T_j$, for all i and j and hence at least one of the induced edge values is a non-triangular number, which is a contradiction.
Case 2. $f(v_1) = 0$.
Then the two adjacent vertices of v_1 must be assigned the triangular numbers 1 and T_i, $i > 1$, again leading to a contradiction.

III. EMBEDDINGS OF TRIANGULAR SUM GRAPHS

In this section we show that the complete graphs K_3 and K_4 can be embedded as induced sub graphs of triangular sum graphs. We also prove that the dutch windmill DW (*n*) can be embedded as an induced sub graph of a triangular sum graph.
Theorem 3.1. Let G be a unicyclic graph consisting of a unique triangle (v_1,v_2,v_3,v_1) with $\deg(v_2) = \deg(v_3) = 2$, a path $P = v_1,u_1,u_2,\dots,u_n$ of length n and k pendant vertices $w_1,w_2,\dots,w_k$ adjacent to v_1. Then G is a triangular sum graph for all $n \ge 3$.
Proof. Define $f:V(G) \to N$ by

$$f(v_1) = 0,$$
$$f(v_i) = T_{2i-1},\ i = 2,3.$$
$$f(u_1) = 1$$
$$f(u_i) = T_{2(i-1)} - f(u_{i-1}),\ i = 2,3.$$
$$f(u_i) = T_{i+3} - f(u_{i-1}),\ 4 \le i \le n \text{ and}$$
$$f(w_i) = T_{n+3+i},\ 1 \le i \le k.$$

Clearly $f+(G) = \{T1, T2, \dots, Tq\}$and hence G is a triangular sum graph.
Lemma 3.2. The complete graph K_4 can be embedded as an induced subgraph of a triangular sum graph.
Proof. Let $V(K_4)=\{v_1, v_2, v_3, v_4\}$. Let G be the graph obtained by attaching 40 pendant vertices v_i, $4 \le i \le 43$ to v_1. Then f defined by $f(v_1) = 0$, $f(v_2) = T_9 = 45$, $f(v_3) = T_{13} = 91$, $f(v_4) = T_{16} = 990$ and $f(v_5,v_6,\dots,v_{43}) = \{T_1,T_2,\dots,T_{43}\}-\{T_9,T_{13},T_{16}\}$ is a triangular sum labeling of G.

It is checked, with the help of a computer, that there are 446 triangular numbers within the range $[1,10^5]$. Also we found that there does not exist 4 triangular numbers, which on pairwise addition gives triangular numbers. Hence we strongly believe that the complete graph K_5 is a forbidden subgraph for a triangular sum graph.
Conjecture 3.3. The complete graph K_n, $n \ge 5$ is a forbidden subgraph for a triangular sum graph.
Theorem 3.4. The dutch windmill DW(n) can be embedded as an induced subgraph of a triangular sum graph.
Proof. Denote the root vertex of the dutch windmill DW(n) as u and the remaining two vertices of i^{th} triangle as v_{2i-1} and v_{2i} for $1 \le i \le n$ respectively. Consider the equation

$$i^2+j^2+i+j-k-k^2 = 24 \qquad (1)$$

Let (x_i, y_i, z_i), where $x_i < y_i < z_i, 1 \le i \le n$ be the first n solutions of equation (1) [viz. (6, 8, 9),(7, 15, 16), (10, 20, 22) etc.].
Define,

$$f(u) = 6,$$
$$f(v_{2i-1}) = T_{x_i} - 6\ \ 1 \le i \le$$

$$f(v_{2i}) = T_{3i} - 6, \quad 1 \leq i \leq n$$

Then one can see that the vertices u and v1 receive triangular numbers and all the remaining vertices receive non-triangular numbers by the above map f. Join two pendant vertices of K1, b, b = zn-3n, to the vertices u and v1. Without loss of generality, assign the number 0 to the root vertex and the missing triangular numbers from T1 to T_{z_n} to the pendant vertices of K1,b in a one-to-one manner. Clearly the induced edge values of the resulting graph is {T1,T2,…, T_{z_n} }. Hence the proof.

Given below is an example for embedding of DW (2) as an induced subgraph of a triangular sum graph.

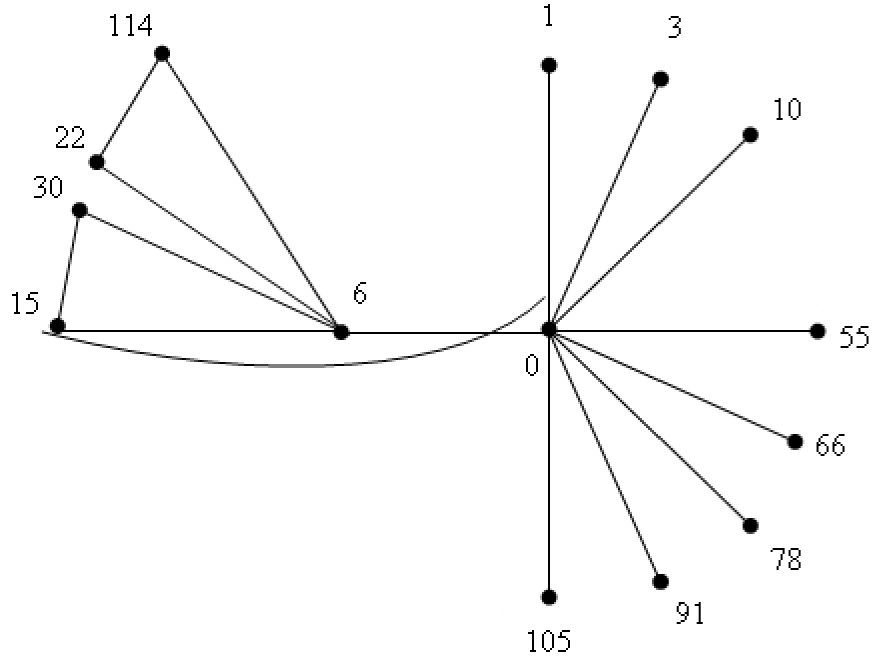

Fig. 5: Embedding of DW (2) as an induced subgraph of a triangular su

IV. CONCLUSIONS

As every graph cannot be embedded as an induced subgraph of a triangular sum graph, it is interesting to embed classes of graphs as an induced subgraph of a triangular sum graph. Also, we would like to study more on cycles admitting triangular sum labeling.

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