Some Properties of Bipolar Fuzzy Normal HX Subgroup and its Normal Level Sub HX Groups

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Abstract

In this paper, the notion of the normal level sub HX group is introduced. We discuss the properties of normal level sub HX groups of a bipolar fuzzy normal HX subgroup of a HX group under homomorphism and anti-homomorphism. AMS Subject Classification (2000): 20N25, 03E72, 03F055, 06F35, 03G25.

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I. INTRODUCTION

In 1965, Zadeh [23] introduced the notion of fuzzy subsets as a generalization of the notion of subsets in ordinary set theory. There were several attempts to fuzzily various mathematical structures. The fuzzifications of algebraic structures was initiated by Rosenfeld [21]. He introduced the notion of fuzzy subgroups and obtained some of their basic properties. Most of the recent works on fuzzy groups follow Rosenfeld's definition. S.D.Kim et.al [8] investigated the concepts of Holomorphic images and preimage on fuzzy ideal. Homomorphic images and pre image of anti-fuzzy ideals are investigated by K.H.Him et.al [9]. The notion of anti-holomorphic images and pre image of fuzzy and anti-fuzzy ideals are investigated by K.C.Chandrasekara Rao et.al [3]. The concept of HX group was introduced by Li Hongxing [12] and the authors Luo Chengzhong, Mi Honghai, Li Hong Xing [13] introduced the concept of fuzzy HX group. In fuzzy set the membership degree of elements ranges over the interval [0, 1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Bipolar valued fuzzy set, which was introduced by K.M.Lee [11] are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [□1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [□1, 0) indicates that elements somewhat satisfy the implicit counter-property.S.V.Manemaran et.al [14] introduced the concept of bipolar fuzzy groups and fuzzy d-ideals of group under (T,S) norm and investigate several properties. R.Muthuraj et.al [17] introduced the concept of bipolar fuzzy HX subgroup and its level sub HX groups.R.Muthuraj et.al [18] introduced the concept of bipolar fuzzy normal HX subgroup. In this paper we discuss the properties of level sub HX groups under homomorphism and anti-homomorphism and also normal level sub HX groups under homomorphism and anti-homomorphism.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G,*) is a finite group, e is the identity element of G, and xy we mean x * y.

A. Definition 2.1 [23]

Let S be a non-empty set. A fuzzy set μ on S is a function $\mu: S \to [0,1]$.

B. Definition 2.2

Let f be any function from a set S to a set T, and let μ be any fuzzy subset of S . Then μ is called f-invariant if f(x) = f(y) implies $\mu(x) = \mu(y)$, where $x,y \in S$.

C. Definition 2.3

A fuzzy subset μ of X is said to have sup property if, for any subset A of X, there exist $a_0 \in A$ such that $\mu(a_0) = \max\{\mu(a) ; a \in A\}$.

D. Definition 2.4

A fuzzy subset μ of X is said to have inf property if, for any subset A of X, there exist $a_0 \in A$ such that $\mu(a_0) = \min\{\mu(a) ; a \in A\}$.

E. Definition 2.5 [13]

Let G be a finite group. In $2^G - \{\phi\}$, a non-empty set $\vartheta \subset 2^G - \{\phi\}$ is called a HX group of G, if ϑ is a group with respect to the algebraic operation defined by $AB = \{ab / a \in A \text{ and } b \in B\}$.

F. Definition 2.6 [16]

A mapping f from a HX group ϑ_1 to a HX group ϑ_2 is said to be a homomorphism if $f(XY) = f(X) f(Y)$ for all $X,Y \in \vartheta_1$.

G. Definition 2.7 [16]

A mapping f from a HX group ϑ_1 to a HX group ϑ_2 (ϑ_1 and ϑ_2 are not necessarily commutative) is said to be an anti-homomorphism if $f(XY) = f(Y) f(X)$ for all $X,Y \in \vartheta_1$.

H. Definition 2.8 [17]

Let ϑ be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set λ_μ in ϑ is an object having the form $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ where $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$ are mappings. The positive membership degree $\lambda_\mu^+(A)$ denotes the satisfaction degree of an element A to the property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ and the negative membership degree $\lambda_\mu^-(A)$ denotes the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. If $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) = 0$, it is the situation that A is regarded as having only positive satisfaction for $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. If $\lambda_\mu^+(A) = 0$ and $\lambda_\mu^-(A) \neq 0$, it is the situation that A does not satisfy the property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$, but somewhat satisfies the counter property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$.It is possible for an element A to be such that $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) \neq 0$ when the membership function of property overlaps that its counter property over some portion of ϑ. For the sake of simplicity, we shall use the symbol $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ for the bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$.

I. Definition 2.9

Let f be any function from a set ϑ_1 to a set ϑ_2, and let λ_μ be any bipolar fuzzy subset of X. Then λ_μ is called f-invariant if $f(X) = f(Y)$ implies $\lambda_\mu^+(X) = \lambda_\mu^+(Y)$ and $\lambda_\mu^-(X) = \lambda_\mu^-(Y)$,where $X,Y \in \vartheta_1$.

J. Definition 2.10 [16]

Let G_1 and G_2 be any two groups. Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets in G_1 and G_2 respectively. Let $\vartheta_1 \subset 2^{G_1} - \{\phi\}$ and $\vartheta_2 \subset 2^{G_2} - \{\phi\}$ are HX groups defined on G_1 and G_2 respectively. let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar fuzzy subsets defined on ϑ_1 and ϑ_2 respectively induced by μ and φ. Let $f : \vartheta_1 \to \vartheta_2$ be a mapping , then the image $f(\lambda_\mu)$ of λ_μ is the bipolar fuzzy subset $f(\lambda_\mu) = ((f(\lambda_\mu))^+, (f(\lambda_\mu))^-)$ of ϑ_2 defined by for all $f(X) = Y \in \vartheta_2$,where $X \in \vartheta_1$

$$(f(\lambda_\mu))^+(f(X)) = \begin{cases} \max\{(\lambda_\mu)^+(X) : X \in f^{-1}(Y)\}, & \text{if } f^{-1}(Y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

and

$$(f(\lambda_\mu))^-(f(X)) = \begin{cases} \max\{(\lambda_\mu)^-(X) : X \in f^{-1}(Y)\}, & \text{if } f^{-1}(Y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

also the pre-image $f^{-1}(\sigma_\varphi)$ of σ_φ under f is a bipolar fuzzy subset of ϑ_1 defined by $(f^{-1}(\sigma_\varphi))^+(X) = \sigma_\varphi^+(f(X))$, $(f^{-1}(\sigma_\varphi))^-(X) = \sigma_\varphi^-(f(X))$.

K. Definition 2.11 [17]

Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. A bipolar fuzzy set λ_μ defined on ϑ is said to be a bipolar fuzzy subgroup induced by μ on ϑ or a bipolar fuzzy HX subgroup of ϑ if for $X, Y \in \vartheta$,

1) $\lambda_\mu^+(XY) \geq \min\{\lambda_\mu^+(X), \lambda_\mu^+(Y)\}$
2) $\lambda_\mu^-(XY) \leq \max\{\lambda_\mu^-(X), \lambda_\mu^-(Y)\}$
3) $\lambda_\mu^+(X^{-1}) = \lambda_\mu^+(X)$, $\lambda_\mu^-(X^{-1}) = \lambda_\mu^-(X)$, where
   $\lambda_\mu^+(X) = \max\{\mu^+(x) / \text{for all } x \in X \subseteq G\}$ and

$\lambda_{\mu^-}(X) = \min\{\mu^-(x) / \text{for all } x \in X \subseteq G\}$.

L. Definition 2.12 [17]

Let λ_μ be a bipolar fuzzy HX subgroup of a HX group ϑ. The set $U(\lambda_\mu; \alpha, \beta) = \{ A \in \vartheta / \lambda_{\mu^+}(A) \geq \alpha, \lambda_{\mu^-}(A) \leq \beta \}$, for any $< \alpha, \beta > \in [0,1]\times[-1,0]$ is called the bipolar level subset of λ_μ or $< \alpha, \beta >$ - level subset of λ_μ or upper level subset of λ_μ. or level subset of λ_μ. .

M. Theorem 2.13 [17]

Let λ_μ be a bipolar fuzzy HX subgroup of HX group ϑ then for $< \alpha, \beta > \in [0,1]\times[-1,0]$ such that $\lambda_{\mu^+}(E) \geq \alpha$, $\lambda_{\mu^-}(E) \leq \beta$ and $U(\lambda_\mu; \alpha, \beta)$ is a sub HX group of ϑ.

N. Theorem 2.14 [17]

Let ϑ be a HX group and λ_μ be a bipolar fuzzy subset of ϑ such that $U(\lambda_\mu; \alpha, \beta)$ is a sub HX group of ϑ for $< \alpha, \beta > \in [0,1]\times[-1,0]$ such that $\lambda_{\mu^+}(E) \geq \alpha$ and $\lambda_{\mu^-}(E) \leq \beta$, then λ_μ is a bipolar fuzzy sub HX group of ϑ.

O. Definition 2.15 [17]

Let λ_μ be a bipolar fuzzy HX subgroup of a HX group ϑ. The sub HX groups $U(\lambda_\mu; \alpha, \beta)$ for $< \alpha, \beta > \in [0,1]\times[-1,0]$ and $\lambda_{\mu^+}(E) \geq \alpha$, $\lambda_{\mu^-}(E) \leq \beta$ are called bipolar level sub HX groups of λ_μ or level sub HX groups of λ_μ.

P. Definition 2.16 [18]

Let G be a group. Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_{\mu^+}, \lambda_{\mu^-})$ be a bipolar fuzzy subset on ϑ,then λ_μ is said to be bipolar fuzzy normal HX subgroup on ϑ, if

$\lambda_{\mu^+}(XYX^{-1}) \geq \lambda_{\mu^+}(Y)$, $\lambda_{\mu^-}(XYX^{-1}) \leq \lambda_{\mu^-}(Y)$, for all $X, Y \in \vartheta$, where,

$\lambda_{\mu^+}(X) = \max\{\mu^+(x) / \text{for all } x \in X \subseteq G\}$ and

$\lambda_{\mu^-}(X) = \min\{\mu^-(x) / \text{for all } x \in X \subseteq G\}$.

Q. Theorem 2.17 [18]

Let G be a group and $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_{\mu^+}, \lambda_{\mu^-})$ be a bipolar fuzzy HX subgroup of a HX group ϑ, then the following conditions are equivalent

1) λ_μ is a bipolar fuzzy normal HX subgroup on ϑ.
2) $\lambda_\mu(XYX^{-1}) = \lambda_\mu(Y)$, for all $X, Y \in \vartheta$.
3) $\lambda_\mu(XY) = \lambda_\mu(YX)$, for all $X, Y \in \vartheta$.

R. Remark 2.18

Let G be a group. Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G. If $\lambda_\mu = (\lambda_{\mu^+}, \lambda_{\mu^-})$ be a bipolar fuzzy normal HX subgroup of ϑ with $|X| \geq 2$ for all $X \in \vartheta$, then $\mu = (\mu^+, \mu^-)$ need not be a bipolar fuzzy subgroup of G. This can be illustrated by the following Example.

Let $G = \{Z_7 - \{0\}, \bullet_7\}$ be a group. For all $x \in G$, Define a bipolar fuzzy set $\mu = (\mu^+, \mu^-)$ on G as,

x	*1*	*2*	*3*	*4*	*5*	*6*
$\mu^+(x)$	*0.8*	*0.6*	*0.5*	*0.6*	*0.5*	*0.4*
$\mu^-(x)$	*−0.7*	*−0.6*	*−0.4*	*−0.6*	*−0.4*	*−0.3*

But, $\mu^+(2 \cdot 3) = \mu^+(6) = 0.4$

$\text{Min}\{\mu^+(2), \mu^+(3)\} = \min\{0.6, 0.5\} = 0.5$

Hence, $\mu^+(2 \cdot 3) \geq \min\{\mu^+(2), \mu^+(3)\}$

And $\mu^-(2 \cdot 3) = \mu^-(6) = -0.3$

$\text{Max}\{\mu^-(2), \mu^-(3)\} = \max\{-0.6, -0.4\} = -0.4$

Hence, $\mu^-(2 \cdot 3) \leq \max\{\mu^+(2), \mu^+(3)\}$

So, μ is not a bipolar fuzzy subgroup of G.

Let $\vartheta = \{\{1,6\},\{2,5\},\{3,4\}\}$, then $(\vartheta, \bullet_7)$ be a HX subgroup of G. where $E = \{1,6\}$, $A = \{2,5\}$, $B = \{3,4\}$.

$\bullet_7$	*E*	*A*	*B*
E	*E*	*A*	*B*
A	*A*	*B*	*E*
B	*B*	*E*	*A*

For all $X \in \vartheta$, Define a bipolar fuzzy set $\lambda_\mu = (\lambda_{\mu^+}, \lambda_{\mu^-})$ on ϑ as,

$$\lambda_{\mu^+}(X) = \max\{\mu^+(x) / \text{for all } x \in X \subseteq G\}$$
$$\lambda_{\mu^-}(X) = \min\{\mu^-(x) / \text{for all } x \in X \subseteq G\}$$

Now,

$\lambda_{\mu^+}(E) = \lambda_{\mu^+}(\{1,6\}) = \max\{\mu^+(1), \mu^+(6)\} = \max\{0.8, 0.4\} = 0.8$

$\lambda_\mu^+(A) = \lambda_\mu^+(\{2,5\}) = \max\{\mu^+(2), \mu^+(5)\} = \max\{0.6, 0.5\} = 0.6$
$\lambda_\mu^+(B) = \lambda_\mu^+(\{3,4\}) = \max\{\mu^+(3), \mu^+(4)\} = \max\{0.5, 0.6\} = 0.6$
$\lambda_\mu^-(E) = \lambda_\mu^-(\{1,6\}) = \min\{\mu^-(1), \mu^-(6)\} = \min\{-0.7, -0.3\} = -0.7$
$\lambda_\mu^-(A) = \lambda_\mu^-(\{2,5\}) = \min\{\mu^-(2), \mu^-(5)\} = \min\{-0.6, -0.4\} = -0.6$
$\lambda_\mu^-(B) = \lambda_\mu^-(\{3,4\}) = \min\{\mu^-(3), \mu^-(4)\} = \min\{-0.4, -0.6\} = -0.6$
and also, $\lambda_\mu^+(XY) = \lambda_\mu^+(YX)$; $\lambda_\mu^-(XY) = \lambda_\mu^-(YX)$, for all $X, Y \in \vartheta$.
By routine calculations, it is easy to see that λ_μ is a bipolar fuzzy normal HX subgroup of ϑ.

S. Theorem 2.19

Let G be a group and $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy normal HX subgroup of a HX group ϑ, then $H = \{X \in \vartheta / \lambda_\mu^+(X) = \lambda_\mu^+(E), \lambda_\mu^-(X) = \lambda_\mu^-(E)\}$ is a normal sub HX group of ϑ.

1) Proof

Let $H = \{X \in \vartheta / \lambda_\mu^+(X) = \lambda_\mu^+(E), \lambda_\mu^-(X) = \lambda_\mu^-(E)\}$.
Let H be a non-empty subset of ϑ, since $E \in H$.
By Theorem 3.3 (ii) [17] , H is a sub HX group of ϑ.
For any $A \in \vartheta$ and $X \in H$,
Now, $\lambda_\mu^+(AXA^{-1}) = \lambda_\mu^+(A^{-1}AX)$
$= \lambda_\mu^+(X)$
$\lambda_\mu^+(AXA^{-1}) = \lambda_\mu^+(X)$.
and $\lambda_\mu^-(AXA^{-1}) = \lambda_\mu^{-+}(A^{-1}AX)$
$= \lambda_\mu^-(X)$
$\lambda_\mu^-(AXA^{-1}) = \lambda_\mu^-(X)$.
Hence, H is a normal sub HX group of ϑ.

T. Theorem 2.20 [18]

Let G be a group. Let λ_μ be a bipolar fuzzy HX subgroup of a HX group ϑ is a bipolar fuzzy normal HX subgroup on ϑ if and only if for any $<\alpha,\beta> \in [0,1]\times[-1,0]$, $U(\lambda_\mu ; \alpha, \beta)$ is a normal sub HX group of ϑ.

U. Definition 2.21 [18]

Let λ_μ be a bipolar fuzzy normal HX subgroup of a HX group ϑ. The normal sub HX groups $U(\lambda_\mu ; \alpha, \beta)$, for $<\alpha, \beta> \in [0,1] \times [-1,0]$ and $\lambda_\mu^+(E) \geq \alpha$, $\lambda_\mu^-(E) \leq \beta$ are called normal level sub HX groups of λ_μ.

III. PROPERTIES OF LEVEL SUB HX GROUPS OF BIPOLAR FUZZY HX SUBGROUP OF A HX GROUP UNDER HOMOMORPHISM AND ANTI-HOMOMORPHISM

In this section, we investigate the properties of level sub HX groups of bipolar fuzzy HX subgroup of a HX group under homomorphism and anti-homomorphism.

A. Theorem 3.1

Let ϑ_1 and ϑ_2 be any two HX groups. f: $\vartheta_1 \to \vartheta_2$ be a mapping and $U(\lambda_\mu ; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup of ϑ_1. Then $U(f(\lambda_\mu) ; \alpha, \beta) = f(U(\lambda_\mu ; \alpha, \beta))$.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f : \vartheta_1 \to \vartheta_2$ be a mapping.
Let $U(\lambda_\mu ; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup of ϑ_1.
Let $X \in U(\lambda_\mu ; \alpha, \beta)$ such that $f(X) \in U(f(\lambda_\mu) ; \alpha, \beta)$,where $<\alpha, \beta> \in [0,1]\times[-1,0]$.
Let $f(X) \in U(f(\lambda_\mu); \alpha, \beta) \Leftrightarrow (f(\lambda_\mu))^+ f(X) \geq \alpha$ and $(f(\lambda_\mu))^- f(X) \leq \beta$
$\Leftrightarrow \lambda_\mu^+(X) \geq \alpha$ and $\lambda_\mu^-(X) \leq \beta$
$\Leftrightarrow X \in U(\lambda_\mu ; \alpha, \beta)$
$\Leftrightarrow f(X) \in f(U(\lambda_\mu ; \alpha, \beta))$
Hence, $U(f(\lambda_\mu); \alpha, \beta) = f(U(\lambda_\mu ; \alpha, \beta))$.

B. Theorem 3.2

Let ϑ_1 and ϑ_2 be any two HX groups. Let $f : \vartheta_1 \to \vartheta_2$ be a mapping and $U(\sigma_\varphi ; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup of ϑ_2. Then $U((f^{-1}(\sigma_\varphi) ; \alpha, \beta) = f^{-1}(U(\sigma_\varphi ; \alpha, \beta))$

2) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f : \vartheta_1 \to \vartheta_2$ be a mapping.
Let $U(\sigma_\varphi ; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup of ϑ_2.
Let $X \in U(f^{-1}(\sigma_\varphi); \alpha, \beta) \Leftrightarrow (f^{-1}(\sigma_\varphi))^+(X) \geq \alpha$ and $(f^{-1}(\sigma_\varphi))^-(X) \leq \beta$
$\Leftrightarrow \sigma_\varphi{}^+(f(X)) \geq \alpha$ and $\sigma_\varphi{}^-(f(X)) \leq \beta$
$\Leftrightarrow f(X) \in U(\sigma_\varphi ; \alpha, \beta)$
$\Leftrightarrow X \in f^{-1}(U(\sigma_\varphi ; \alpha, \beta))$
Hence, $U(f^{-1}(\sigma_\varphi) ; \alpha, \beta) = f^{-1}(U(\sigma_\varphi ; \alpha, \beta))$.

C. Theorem 3.3

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let λ_μ be a bipolar fuzzy HX subgroup on ϑ_1. If $f: \vartheta_1 \to \vartheta_2$ is a homomorphism and onto, then the image of a level sub HX group $U(\lambda_\mu ; \alpha, \beta)$ of a bipolar fuzzy HX subgroup λ_μ of a HX group ϑ_1 is a level sub HX group $U(f(\lambda_\mu); \alpha, \beta)$ of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.

3) Proof

Let G_1 and G_2 be any two groups and $f: \vartheta_1 \to \vartheta_2$ be a homomorphism.
Let λ_μ be a bipolar fuzzy HX subgroup of a HX group ϑ_1.
Clearly, $f(\lambda_\mu)$ is a bipolar fuzzy HX subgroup of a HX group ϑ_2.
Let X and Y in ϑ_1, implies f(X) and f(Y) in ϑ_2.
Let $U(\lambda_\mu ; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup λ_μ of a HX group ϑ_1.
Choose α and β in such a way that $X, Y \in U(\lambda_\mu ; \alpha, \beta)$ and hence $XY^{-1} \in U(\lambda_\mu ; \alpha, \beta)$.
Then, $\lambda_\mu{}^+(X) \geq \alpha$ and $\lambda_\mu{}^-(X) \leq \beta$ and $\lambda_\mu{}^+(Y) \geq \alpha$ and $\lambda_\mu{}^-(Y) \leq \beta$.
Also $\lambda_\mu{}^+(XY^{-1}) \geq \alpha$ and $\lambda_\mu{}^-(XY^{-1}) \leq \beta$.
We have to prove that $U(f(\lambda_\mu); \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.
Now, Let $f(X), f(Y) \in U(f(\lambda_\mu); \alpha, \beta)$.
$(f(\lambda_\mu))^+(f(X)) = \lambda_\mu{}^+(X) \geq \alpha$, implies that $(f(\lambda_\mu))^+(f(X)) \geq \alpha$
$(f(\lambda_\mu))^+(f(Y)) = \lambda_\mu{}^+(Y) \geq \alpha$, implies that $(f(\lambda_\mu))^+(f(Y)) \geq \alpha$ and
$(f(\lambda_\mu))^-(f(X)) = \lambda_\mu{}^-(X) \leq \beta$, implies that $(f(\lambda_\mu))^-(f(X)) \leq \beta$
$(f(\lambda_\mu))^-(f(Y)) = \lambda_\mu{}^-(Y) \leq \beta$, implies that $(f(\lambda_\mu))^-(f(Y)) \leq \beta$.
Let,
$(f(\lambda_\mu))^+(f(X)(f(Y))^{-1}) = (f(\lambda_\mu))^+(f(X)f(Y^{-1}))$, as f is a homomorphism
$= (f(\lambda_\mu))^+(f(XY^{-1}))$, as f is a homomorphism
$= \lambda_\mu{}^+(XY^{-1})$
$(f(\lambda_\mu))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and
$(f(\lambda_\mu))^-(f(X)(f(Y))^{-1}) = (f(\lambda_\mu))^-(f(X)f(Y^{-1}))$, as f is a homomorphism
$= (f(\lambda_\mu))^-(f(XY^{-1}))$, as f is a homomorphism
$= \lambda_\mu{}^-(XY^{-1})$
$\leq \beta$
That is, $(f(\lambda_\mu))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and $(f(\lambda_\mu))^-(f(X)(f(Y))^{-1}) \leq \beta$.
Hence, $f(X)(f(Y))^{-1} \in U(f(\lambda_\mu); \alpha, \beta)$.
Hence, $U(f(\lambda_\mu) ; \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.

D. Theorem 3.4

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let σ_φ be a bipolar fuzzy HX subgroup on ϑ_2. If $f: \vartheta_1 \to \vartheta_2$ is a homomorphism , then the pre- image of a level sub HX group $U(\sigma_\varphi ; \alpha, \beta)$ of a bipolar fuzzy HX subgroup σ_φ of a HX group ϑ_2 is a level sub HX group $U(f^{-1}(\sigma_\varphi) ; \alpha, \beta)$ of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f: \vartheta_1 \to \vartheta_2$ be a homomorphism.
Let σ_φ be a bipolar fuzzy HX subgroup of ϑ_2,
Clearly, $f^{-1}(\sigma_\varphi)$ is a bipolar fuzzy HX subgroup of ϑ_1.
Let X and Y in ϑ_1, implies f(X) and f(Y) in ϑ_2.
Let $U(\sigma_\varphi ; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup σ_φ of a HX group ϑ_2.
Choose α and β in such a way that $f(X), f(Y) \in U(\sigma_\varphi ; \alpha, \beta)$ and hence $f(X)(f(Y))^{-1} \in U(\sigma_\varphi ; \alpha, \beta)$.

Then $\sigma_\varphi^+(f(X)) \geq \alpha$ and $\sigma_\varphi^-(f(X)) \leq \beta$ and $\sigma_\varphi^+(f(Y)) \geq \alpha$ and $\sigma_\varphi^-(f(Y)) \leq \beta$.
Also $\sigma_\varphi^+(f(X)(f(Y))^{-1}) \geq \alpha$ and $\sigma_\varphi^-(f(X)(f(Y))^{-1}) \leq \beta$.
We have to prove that $U(f^{-1}(\sigma_\varphi);\ \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.
Now, Let $X, Y \in U(f^{-1}(\sigma_\varphi);\ \alpha, \beta)$.
$(f^{-1}(\sigma_\varphi))^+(X) = \sigma_\varphi^+(f(X)) \geq \alpha$, implies that $(f^{-1}(\sigma_\varphi))^+(X) \geq \alpha$
$(f^{-1}(\sigma_\varphi))^+(Y) = \sigma_\varphi^+(f(Y)) \geq \alpha$, implies that $(f^{-1}(\sigma_\varphi))^+(Y) \geq \alpha$ and
$(f^{-1}(\sigma_\varphi))^-(X) = \sigma_\varphi^-(f(X)) \leq \beta$, implies that $(f^{-1}(\sigma_\varphi))^-(X) \leq \beta$
$(f^{-1}(\sigma_\varphi))^-(Y) = \sigma_\varphi^-(f(Y)) \leq \beta$, implies that $(f^{-1}(\sigma_\varphi))^-(Y) \leq \beta$.
Let,
$(f^{-1}(\sigma_\varphi))^+(XY^{-1}) = \sigma_\varphi^+(f(XY^{-1}))$, as f is a homomorphism
$= \sigma_\varphi^+(f(X)\, f(Y^{-1}))$, as f is a homomorphism
$= \sigma_\varphi^+(f(X)(f(Y))^{-1})$
$\geq \alpha$
$(f^{-1}(\sigma_\varphi))^+(XY^{-1}) \geq \alpha$ and
$(f^{-1}(\sigma_\varphi))^-(XY^{-1}) = \sigma_\varphi^-(f(XY^{-1}))$, as f is a homomorphism
$= \sigma_\varphi^-(f(X)\, f(Y^{-1}))$, as f is a homomorphism
$= \sigma_\varphi^-(f(X)(f(Y))^{-1})$
$\leq \beta$
$(f^{-1}(\sigma_\varphi))^+(XY^{-1}) \leq \beta$
That is, $(f^{-1}(\sigma_\varphi))^+(XY^{-1}) \geq \alpha$ and $(f^{-1}(\sigma_\varphi))^+(XY^{-1}) \leq \beta$.
Hence, $XY^{-1} \in U(f^{-1}(\sigma_\varphi);\ \alpha, \beta)$
Hence, $U(f^{-1}(\sigma_\varphi);\ \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.

E. Theorem 3.5

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let λ_μ be a bipolar fuzzy HX subgroup on ϑ_1. If f: $\vartheta_1 \rightarrow \vartheta_2$ is an anti-homomorphism and onto, then the image of a level sub HX group $U(\lambda_\mu;\ \alpha, \beta)$ of a bipolar fuzzy HX subgroup λ_μ of a HX group ϑ_1 is a level sub HX group $U(f(\lambda_\mu);\ \alpha, \beta)$ of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.

1) Proof

Let G_1 and G_2 be any two groups and f: $\vartheta_1 \rightarrow \vartheta_2$ be an anti-homomorphism.
Let λ_μ be a bipolar fuzzy HX subgroup of a HX group ϑ_1.
Clearly, $f(\lambda_\mu)$ is a bipolar fuzzy HX subgroup of a HX group ϑ_2.
Let X and Y in ϑ_1, implies f(X) and f(Y) in ϑ_2.
Let $U(\lambda_\mu;\ \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup λ_μ of a HX group ϑ_1.
Choose α and β in such a way that $X, Y \in U(\lambda_\mu;\ \alpha, \beta)$ and hence $Y^{-1}X \in U(\lambda_\mu;\ \alpha, \beta)$.
then $\lambda_\mu^+(X) \geq \alpha$ and $\lambda_\mu^-(X) \leq \beta$, $\lambda_\mu^+(Y) \geq \alpha$ and $\lambda_\mu^-(Y) \leq \beta$.
Also $\lambda_\mu^+(Y^{-1}X) \geq \alpha$ and $\lambda_\mu^-(Y^{-1}X) \leq \beta$.
We have to prove that $U(f(\lambda_\mu);\ \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.
Now, Let $f(X), f(Y) \in U(f(\lambda_\mu);\ \alpha, \beta)$.
$(f(\lambda_\mu))^+(f(X)) = \lambda_\mu^+(X) \geq \alpha$, implies that $(f(\lambda_\mu))^+(f(X)) \geq \alpha$
$(f(\lambda_\mu))^+(f(Y)) = \lambda_\mu^+(Y) \geq \alpha$, implies that $(f(\lambda_\mu))^+(f(Y)) \geq \alpha$ and
$(f(\lambda_\mu))^-(f(X)) = \lambda_\mu^-(X) \leq \beta$, implies that $(f(\lambda_\mu))^-(f(X)) \leq \beta$
$(f(\lambda_\mu))^-(f(Y)) = \lambda_\mu^-(Y) \leq \beta$, implies that $(f(\lambda_\mu))^-(f(Y)) \leq \beta$.
Let,
$(f(\lambda_\mu))^+(f(X)(f(Y))^{-1}) = (f(\lambda_\mu))^+(f(X)f(Y^{-1}))$, as f is an anti-homomorphism
$= (f(\lambda_\mu))^+(f(Y^{-1}X))$, as f is an anti-homomorphism
$= \lambda_\mu^+(Y^{-1}X)$
$(f(\lambda_\mu))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and
$(f(\lambda_\mu))^-(f(X)(f(Y))^{-1}) = (f(\lambda_\mu))^-(f(X)f(Y^{-1}))$, as f is an anti-homomorphism
$= (f(\lambda_\mu))^-(f(Y^{-1}X))$, as f is an anti-homomorphism
$= \lambda_\mu^-(Y^{-1}X)$
$\leq \beta$
That is, $(f(\lambda_\mu))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and $(f(\lambda_\mu))^-(f(X)(f(Y))^{-1}) \leq \beta$.
Hence, $f(X)(f(Y))^{-1} \in U(f(\lambda_\mu);\ \alpha, \beta)$.
Hence, $U(f(\lambda_\mu);\ \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.

F. Theorem 3.6

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let σ_φ be a bipolar fuzzy HX subgroup on ϑ_2. If f: $\vartheta_1 \to \vartheta_2$ is an anti-homomorphism. Then the pre-image of a level sub HX group $U(\sigma_\varphi ;\ \alpha, \beta)$ of a bipolar fuzzy HX subgroup σ_φ of a HX group ϑ_2 is a level sub HX group $U(f^{-1}(\sigma_\varphi) ;\ \alpha, \beta)$ of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let f: $\vartheta_1 \to \vartheta_2$ be an anti-homomorphism.
Let σ_φ be a bipolar fuzzy HX subgroup of ϑ_2,
Clearly, $f^{-1}(\sigma_\varphi)$ is a bipolar fuzzy HX subgroup of ϑ_1.
Let X and Y in ϑ_1, implies f(X) and f(Y) in ϑ_2.
Let $U(\sigma_\varphi ;\ \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup σ_φ of a HX group ϑ_2.
Choose α and β in such a way that $f(X), f(Y) \in U(\sigma_\varphi ;\ \alpha, \beta)$ and hence $(f(Y))^{-1} f(X) \in U(\sigma_\varphi ;\ \alpha, \beta)$.
then $\sigma_\varphi^{+}(f(X)) \geq \alpha$ and $\sigma_\varphi^{-}(f(X)) \leq \beta$ and $\sigma_\varphi^{+}(f(Y)) \geq \alpha$ and $\sigma_\varphi^{-}(f(Y)) \leq \beta$.
Also $\sigma_\varphi^{+}((f(Y))^{-1} f(X)) \geq \alpha$ and $\sigma_\varphi^{-}((f(Y))^{-1} f(X)) \leq \beta$.
We have to prove that $U(f^{-1}(\sigma_\varphi) ;\ \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.
Now, Let $X, Y \in U(f^{-1}(\sigma_\varphi) ;\ \alpha, \beta)$.
$(f^{-1}(\sigma_\varphi))^{+}(X) = \sigma_\varphi^{+}(f(X)) \geq \alpha$, implies that $(f^{-1}(\sigma_\varphi))^{+}(X) \geq \alpha$
$(f^{-1}(\sigma_\varphi))^{+}(Y) = \sigma_\varphi^{+}(f(Y)) \geq \alpha$, implies that $(f^{-1}(\sigma_\varphi))^{+}(Y) \geq \alpha$ and
$(f^{-1}(\sigma_\varphi))^{-}(X) = \sigma_\varphi^{-}(f(X)) \leq \beta$, implies that $(f^{-1}(\sigma_\varphi))^{-}(X) \leq \beta$
$(f^{-1}(\sigma_\varphi))^{-}(Y) = \sigma_\varphi^{-}(f(Y)) \leq \beta$, impies that $(f^{-1}(\sigma_\varphi))^{-}(Y) \leq \beta$.
Let, $(f^{-1}(\sigma_\varphi))^{+}(XY^{-1}) = \sigma_\varphi^{+}(f(XY^{-1}))$, as f is an anti-homomorphism
$= \sigma_\varphi^{+}(f(Y^{-1})f(X))$, as f is an anti-homomorphism
$= \sigma_\varphi^{+}((f(Y))^{-1}f(X))$
$\geq \alpha$
$(f^{-1}(\sigma_\varphi))^{+}(XY^{-1}) \geq \alpha$ and
$(f^{-1}(\sigma_\varphi))^{-}(XY^{-1}) = \sigma_\varphi^{-}(f(XY^{-1}))$, as f is an anti-homomorphism
$= \sigma_\varphi^{+}(f(Y^{-1})f(X))$, as f is an anti-homomorphism
$= \sigma_\varphi^{+}((f(Y))^{-1}f(X))$
$\leq \beta$
$(f^{-1}(\sigma_\varphi))^{+}(XY^{-1}) \leq \beta$
That is, $(f^{-1}(\sigma_\varphi))^{+}(XY^{-1}) \geq \alpha$ and $(f^{-1}(\sigma_\varphi))^{+}(XY^{-1}) \leq \beta$.
Hence, $XY^{-1} \in U(f^{-1}(\sigma_\varphi) ;\ \alpha, \beta)$.
Hence, $U(f^{-1}(\sigma_\varphi) ;\ \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.

IV. PROPERTIES OF NORMAL LEVEL SUB HX GROUP OF BIPOLAR FUZZY NORMAL HX SUBGROUP OF A HX GROUP UNDER HOMOMORPHISM AND ANTI-HOMOMORPHISM

In this section, we investigate the properties of normal level sub HX groups of bipolar fuzzy normal HX subgroup of a HX group under homomorphism and anti-homomorphism.

A. Theorem 4.1

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let λ_μ be a bipolar fuzzy normal HX subgroup on ϑ_1. If f: $\vartheta_1 \to \vartheta_2$ is a homomorphism and onto, then the image of a normal level sub HX group $U(\lambda_\mu ;\ \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup λ_μ of a HX group ϑ_1 is a normal level sub HX group $U(f(\lambda_\mu);\ \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f : \vartheta_1 \to \vartheta_2$ be a homomorphism.
Let λ_μ be a bipolar fuzzy normal HX subgroup of ϑ_1,
Clearly, $f(\lambda_\mu)$ is a bipolar fuzzy normal HX subgroup of ϑ_2.
Let $U(\lambda_\mu ; \alpha,\beta)$ be a normal level sub HX group of a bipolar fuzzy normal HX subgroup of ϑ_1.
Since f is a homomorphism,
By Theorem 3.3, $U(f(\lambda_\mu) ; \alpha,\beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\mu)$ of ϑ_2 .

Let $X \in \vartheta_1$, $A \in U(\lambda_\mu ;\alpha,\beta)$ then $XAX^{-1} \in U(\lambda_\mu ; \alpha,\beta)$,since $U(\lambda_\mu ; \alpha,\beta)$ is normal level sub HX group.
Let, $f(X) \in \vartheta_2$, $f(A) \in U(f(\lambda_\mu) ; \alpha,\beta)$
Now, $f(X) f(A) (f(X))^{-1} = f(X) f(A) f(X^{-1})$
$= f(XAX^{-1})$
$\in f (U(\lambda_\mu ;\alpha,\beta))$,as f is homomorphism.
Clearly, $f(X) f(A) (f(X))^{-1} \in U(f(\lambda_\mu) ; \alpha,\beta))$. (By Theorem 3.1)
Hence, $U(f(\lambda_\mu) ; \alpha,\beta))$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f(\lambda_\mu)$ of ϑ_2.

B. Theorem 4.2

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let σ_φ be a bipolar fuzzy normal HX subgroup on ϑ_2. If f: $\vartheta_1 \rightarrow \vartheta_2$ is a homomorphism, then the pre-image of a normal level sub HX group $U(\sigma_\varphi ; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup σ_φ of a HX group ϑ_2 is a normal level sub HX group $U(f^{-1} (\sigma_\varphi) ; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $f^{-1} (\sigma_\varphi)$ of a HX group ϑ_1.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f : \vartheta_1 \rightarrow \vartheta_2$ be a homomorphism.
Let σ_φ be a bipolar fuzzy normal HX subgroup of ϑ_2,
Clearly, $f^{-1} (\sigma_\varphi)$ is a bipolar fuzzy normal HX subgroup of ϑ_1.
Let $U(\sigma_\varphi ; \alpha,\beta)$ be a normal level sub HX group of a bipolar fuzzy normal HX subgroup σ_φ of ϑ_2.
Since f is a homomorphism,
By Theorem 3.4, $U(f^{-1} (\sigma_\varphi) ; \alpha , \beta)$ is a level sub HX group.
Let $X \in \vartheta_1$, $A \in U(f^{-1} (\sigma_\varphi) ; \alpha, \beta)$
As, $U(\sigma_\varphi ; \alpha,\beta)$ be a normal level sub HX group then $f(X) \in \vartheta_2$, $f(A) \in U(\sigma_\varphi ; \alpha, \beta)$
Such that, $f(X) f(A) (f(X)^{-1}) \in U(\sigma_\varphi ;\alpha,\beta)$
$= f(X) f(A) f(X^{-1}) \in U(\sigma_\varphi ;\alpha,\beta)$
$= f(XAX^{-1}) \in U(\sigma_\varphi ;\alpha,\beta)$, as f is a homomorphism
$= XAX^{-1} \in f^{-1} U(\sigma_\varphi ; \alpha,\beta)$
Clearly, $XAX^{-1} \in U(f^{-1}(\sigma_\varphi);\alpha,\beta)$. (By Theorem 3.2)
Hence, $U(f^{-1}(\sigma_\varphi);\alpha,\beta)$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f^{-1} (\sigma_\varphi)$ of ϑ_1.

C. Theorem 4.3

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let λ_μ be a bipolar fuzzy normal HX subgroup on ϑ_1. If f: $\vartheta_1 \rightarrow \vartheta_2$ is an anti-homomorphism and onto, then the image of a normal level sub HX group $U(\lambda_\mu ; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup λ_μ of a HX group ϑ_1 is a normal level sub HX group $U(f(\lambda_\mu) ; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $f(\lambda_\mu)$ of a HX group ϑ_2.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f : \vartheta_1 \rightarrow \vartheta_2$ be an anti-homomorphism.
Let λ_μ be a bipolar fuzzy normal HX subgroup of ϑ_1,
Clearly, $f (\lambda_\mu)$ is a bipolar fuzzy normal HX subgroup of ϑ_2.
Let $U(\lambda_\mu ; \alpha,\beta)$ be a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f(\lambda_\mu)$ of ϑ_1.
Since, f is an anti-homomorphism,
By Theorem 3.3, $U(f(\lambda_\mu) ; \alpha,\beta)$ is a level sub HX group of ϑ_2.
Let $X \in \vartheta_1$, $A \in U(\lambda_\mu ;\alpha,\beta)$ then $X^{-1}AX \in U(\lambda_\mu ;\alpha,\beta)$, since $U(\lambda_\mu ; \alpha,\beta)$ be a normal level sub HX group.
Let, $f(X) \in \vartheta_2$, $f(A) \in U(f(\lambda_\mu) ;\alpha,\beta)$
Now, $f(X) f(A) (f(X))^{-1} = f(X) f(A) f(X^{-1})$
$= f(X^{-1}AX)$
$\in f (U(\lambda_\mu ;\alpha,\beta))$,as f is an anti-homomorphism
Clearly, $f(X) f(A) (f(X))^{-1} \in U(f(\lambda_\mu) ;\alpha,\beta)$. (By Theorem 3.1)
Hence, $U(f(\lambda_\mu) ;\alpha,\beta)$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f(\lambda_\mu)$ of ϑ_2.

D. Theorem 4.4

Let G_1 and G_2 be any two groups and ϑ_1 and ϑ_2 be HX groups on G_1 and G_2 respectively. Let σ_φ be a bipolar fuzzy normal HX subgroup on ϑ_2. If f: $\vartheta_1 \rightarrow \vartheta_2$ is an anti-homomorphism and onto, then the pre-image of a normal level sub HX group

$U(\sigma_\varphi ; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup σ_φ of a HX group ϑ_2 is a normal level sub HX group $U(f^{-1}(\sigma_\varphi) ; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $f^{-1}(\sigma_\varphi)$ of a HX group ϑ_1.

1) Proof

Let ϑ_1 and ϑ_2 be any two HX groups.
Let $f: \vartheta_1 \to \vartheta_2$ be an anti-homomorphism.
Let σ_φ be a bipolar fuzzy normal HX subgroup of ϑ_2,
Clearly, $f^{-1}(\sigma_\varphi)$ is a bipolar fuzzy normal HX subgroup of ϑ_1.
Let $U(\sigma_\varphi ; \alpha,\beta)$ be a normal level sub HX group of a bipolar fuzzy normal HX subgroup σ_φ of ϑ_2.
By Theorem 3.4, $U(f^{-1}(\sigma_\varphi) ; \alpha , \beta)$ is a level sub HX group.
Let $X \in \vartheta_1$, $A \in U(f^{-1}(\sigma_\varphi) ; \alpha, \beta)$
As, $U(\sigma_\varphi ; \alpha,\beta)$ be a normal level sub HX group then $f(X) \in \vartheta_2$, $f(A) \in U(\sigma_\varphi ; \alpha, \beta)$
Such that, $(f(X)^{-1})f(A)\, f(X) \in U(\sigma_\varphi ;\alpha,\beta)$
$= f(X^{-1})f(A)\, f(X) \in U(\sigma_\varphi ;\alpha,\beta)$
$= f(XAX^{-1}) \in U(\sigma_\varphi ;\alpha,\beta)$, as f is an anti-homomorphism
$\Rightarrow XAX^{-1} \in f^{-1}(U(\sigma_\varphi ; \alpha,\beta))$
Clearly, $XAX^{-1} \in U(f^{-1}(\sigma_\varphi);\alpha,\beta)$. (By Theorem 3.2)
Hence, $U(f^{-1}(\sigma_\varphi);\alpha,\beta)$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f^{-1}(\sigma_\varphi)$ of ϑ_1.

V. CONCLUSION

In this paper, we introduced the notion of the normal level sub HX group of a bipolar fuzzy normal HX subgroup and studied this structure. Finally, we proved some results on image and preimage under homomorphism and anti-homomorphism on normal level sub HX group of a bipolar fuzzy normal HX subgroup.

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REFERENCES

[1] S.Abou-Zaid, On normal fuzzy subgroups, J.Fac.Ed.Ain Shams Univ. Cairo, 13, 1998, pp. 115- 125.
[2] K.Chandrasekahara Rao, V.Swaminathan, Anti homomorphisms in Near Rings, Journal of Inst.of Maths.Comp.Science (Maths.ser) ,Vol.21 , No.2, 2008, pp.83-88.
[3] K.Chandrasekahara Rao, V.Swaminathan, Anti homomorphisms in Fuzzy Ideals,World Academy of Science ,Engineering and Technology, Vol.4, 2010, pp.08-29.
[4] F.P.Choudhury, A.B.Chakraborty, S.S.SKhare, A note on fuzzy subgroups and fuzzy homomorphism, J.Math.Anal.Appl., 131, 1988, pp. 537-553.
[5] P.S.Das, Fuzzy groups and level subgroups, J.Math.Anal.Appl , 84, 1981, pp.264-269.
[6] V.N. Dixit , A.Rajesh Kumar, Naseem Ajmal , Level subgroups and union of fuzzy subgroups, Fuzzy Sets and Systems, 37, 1990, pp.359-371.
[7] G.Gopalakrishnamoorthy, Ph.D Thesis , Alagappa university, (May2000), Karaikudi, Tamilnadu,India.
[8] S.D.Kim , H.S.Kim, On fuzzy ideals of near rings, Bull.Korean math-Soc., 33, 1996, pp.593-601.
[9] K.H.Kim , Y.B.Jun, On Anti fuzzy Ideals in Near Rings , Iranian Journal of Fuzzy System , Vol.2, No.2, 2005, pp.71-80.
[10] I.J.Kumar, P.K.Saxena , P.Yadav, Fuzzy normal subgroups and fuzzy quotients, Fuzzy Sets and Systems , 46, 1992, pp.121-132.
[11] K.M.Lee , Bipolar –valued fuzzy sets and their operations, Pro.Int.Conf.on Intelligent Technologies, Bangkok, Thailand, 2000, pp. 307-312.
[12] Li Hongxing, HX group, BUSEFAL, 33, 1987, pp.31 – 37.
[13] Luo Chengzhong, Mi Honghai , Li Hongxing, Fuzzy HX group, BUSEFAL, 41-14 , 1989, pp.97 – 106.
[14] S.V.Manemaran , B.Chellappa , Structure on Bipolar fuzzy groups and bipolar fuzzy D-ideals under (T,S) norms, International Journal of Computer Applications, Volume 9, 2010, pp. 7-10.
[15] Mehmet Sait Eroglu, The Homomorphic image of a fuzzy subgroup is always a fuzzy subgroup, Fuzzy Sets and Systems, 33, 1989, pp. 255- 256.
[16] R.Muthuraj , M.Sridharan, Homomorphism and anti-homomorphism on Bipolar fuzzy sub HX groups, Gen.Math.Notes, (ISSN 2219-7184), Volume 17,Number 2, 2013, pp. 53-65.
[17] R.Muthuraj , M.Sridharan, Bipolar fuzzy HX group and its level Sub HX groups, International Journal of Mathematical Archive, (ISSN 2229-5046), 5(1), 2014, pp. 230 - 239.
[18] R.Muthuraj , M.Sridharan, Bipolar fuzzy Normal HX subgroups, International Journal of Engineering Associates, (ISSN 2320-0804), Volume 3, Issue7, 2014, pp.8-13.
[19] Mohamed Asaad, Groups and Fuzzy subgroups, Fuzzy Sets and Systems, 39, 1991, pp.323- 328.
[20] Rajesh Kumar, Homomorphism and fuzzy (fuzzy normal) subgroups, Fuzzy Sets and System, 44 , 1991, pp.165 – 168.
[21] Rosenfeld A, Fuzzy Groups, J.Math. Anal. Appl. 35, 1971, pp. 512-517.
[22] Wu Wangming, Normal Fuzzy Subgroups, Fuzzy Math.(in Chinese),1, 1981, pp. 21-30.
[23] Zadeh L.A, Fuzzy Sets, Inform and Control, 8, 1965, pp. 338-365.
[24] Zhang W.R, Bipolar fuzzy sets, Proc. of FUZZ-IEEE, 1998, pp. 835-840.